Interdisciplinary Lively Application Project
Pendulum Activity

Title: Pendulum Activity

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Course Classification: Calculus 1 (MATH 1401 at CU-Denver),

Disciplinary Classifications: Mathematics, Physics

Prerequisite Skills:

1. Familiarity with Trigonometric Functions
2. Transformation of Functions
3. Differentiation and integration and relationship with distance, velocity, and acceleration
4. Graphing

Mathematics Classifications: Modeling, differentiation, integration, ODE

Physics Classifications: Speed, velocity, acceleration, gravity, Newton's second law of motion.

Materials Required:

1. String (approximately 5 ft for each group)
2. Tape (to tape string to ceiling)
3. 1" washers (one for each group)
4. Stopwatch (one for each group)
5. Tape measure (one for every 2 or 3 groups)
6. Graphing or engineering paper (2 or 3 per person)
7. Scissors (for cutting string).

Interdisciplinary Lively Application Project:

Pendulum Activity

By Bruce MacMillan, Sam Welch, Lynn Bennethum, and Tracy Lawrence

Introduction In the first half of this activity (Part A) we are going to build a simple pendulum using string and a washer. We will use measured data and an assumed sinusoidal functional form to obtain an expression for the motion of the pendulum. We will then apply some calculus concepts to this function and its graph to further understand topics such as velocity, acceleration, etc.

In the second half of the activity (Part B) we will apply Newton's second law of motion to obtain an analytical model for the pendulum. This analytic model will be used to obtain an analytic expression for the same motion parameter as was obtained in the first part of this project.

You will be asked to turn in a report in the following format (although you will be performing the experiments as a group, each individual must turn in his/her own individual report). The report should include

1. A title page with names of group members (up to three in one group).

2. A short introduction containing objectives of the activity.

3. A discussion which includes the task items. This should be written in proper English.

4. A conclusion.

The report may be handwritten or typed. It must be written grammatically correct. It is *not* enough to just turn in equations and graphs with no explanations (this sort of report will receive a failing grade). Graphs and equations should be motivated (why are we doing this?) and an explanation must be provided on how each new equation is derived.

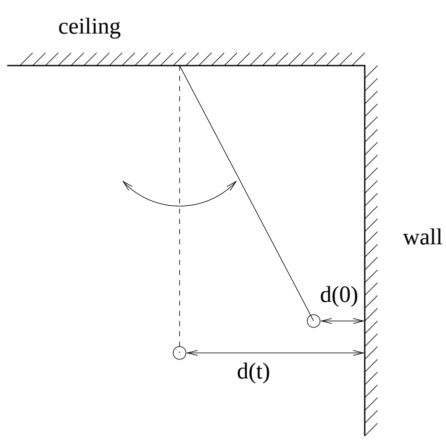

Figure 1: Pendulum Set-Up

Part A
I. Data Collection and Modeling of the Pendulum Motion

The following worksheet is to be used to guide you in collecting data. Be sure to save this information so that you can include it in your written report. You will not be turning in these sheets. First, set up the pendulum as in Figure 1. Tie the washer to one end of the string, and attach the other end to the ceiling (using a tack for example). With the string attached to the ceiling, the washer should be easily pulled within a hands distance to a nearby wall.

For the pendulum in your experiment, you will need to know the following measurements. So measure:

1. The length of the pendulum in centimeters (measure from the ceiling to the middle of the washer). ______cm

2. The distance from the wall to the pendulum when it is at rest. ______cm

You will also need to know the distance from the wall to where the pendulum will be released. (Note: It doesn't matter where you release the pendulum, but you need to know what this value is!). ______cm

Let $d(t)$ be a function that represents the distance (in cm) of the pendulum from the wall with respect to time, t, in seconds. The pendulum will be released at time $t = 0$. The period of the sinusoidal function is the time of one cycle of the pendulum. To evaluate this, release the pendulum (from the position indicated earlier) and find the time of 5 complete cycles of the pendulum. The period of one cycle, of course, will be this time divided by 5. Do this and record the time of 5 cycles. ______s. Try it again to check your answer. You need to have an accurate estimate of the period for the sinusoidal function. (Accurate to the nearest 0.1 seconds.)

The period of one cycle is ______s.

Now with your group sketch a graph (on graph/engineering paper) of the first 2 periods of the function, $d(t)$, which represents the distance from the wall as a function of the time t. (Remember, all graphs and equations should be included in your report.) Assume no damping (i.e. assume the amplitude remains constant). Begin by locating the initial point ($t = 0$) on your graph. Then locate the point when the washer reaches its maximum distance from the wall and label this point on your graph. Continue to plot and label the important points until you have shown two complete periods of the sinusoidal function. Be sure to include this in your report.

From your graph, state the following characteristics of the sinusoidal function.
Vertical shift ______, Amplitude ______, Period ______, Phase Shift ______.

We could use either a cosine or a sine function as a model.

1. Using a sine function will a vertical shift be required? (yes or no) ______.

2. Using a sine function will a horizontal shift be required? (yes or no) ______.

3. Using a sine function will a reflection be required? (yes or no) ______.
4. Using a cosine function will a vertical shift be required? (yes or no) ______.
5. Using a cosine function will a horizontal shift be required? (yes or no) ______.
6. Using a cosine function will a reflection be required? (yes or no) ______.

Using a cosine function and the above information regarding vertical shift, amplitude, period, phase shift, and reflection, determine the function $d(t)$ (round decimals to 0.001).

$$d(t) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \qquad (1)$$

Sketch $d(t)$ on your calculator/computer with a window similar to your graph (by hand) to check your equation. Be sure to print the plot for your report.

II. Using the Model of the Pendulum Motion

Using your $d(t)$ from Part I: How far from the wall is the pendulum **1 second** after you released it? ______.
5 seconds? ______.

Determine the velocity and acceleration functions.

$$v(t) = d'(t) = \rule{8cm}{0.4pt}$$

$$a(t) = v'(t) = d''(t) = \rule{8cm}{0.4pt}$$

(Note: You could check your answer with a computer algebra system).

Plot and print **two periods** of $v(t)$ and $a(t)$. Label the graphs and label the important points. Be sure to include these graphs in your report.

How fast is the pendulum moving 2 seconds after it was released? (include units) ______.
Is it moving **toward** or **away from** the wall at this time? ________________________________
Explain. ____.

Let's use the graphs (the two periods) to answer some questions about the pendulum.

1. Over what time intervals is the pendulum moving toward the wall?
   ____.

2. When is the pendulum moving the fastest? $t =$ ______. Explain:
   ____.

Include these answers in your report.

The **speed** of the pendulum can be represented by the **absolute value of the velocity function.** Plot and print $|v(t)|$. Be sure to include your graph in your report.

The pendulum is speeding up when the "speed function" is **increasing**. Over what time intervals is the pendulum speeding up?

____.

Now let's see what the velocity function and the acceleration functions are doing on these time intervals. Deactivate the "speed function" and activate the velocity and acceleration functions for plotting. Rewrite the intervals above and tell whether the velocity and acceleration functions are **positive** or **negative** at these times.

Interval ____________	Velocity ____________	Acceleration ____________
Interval ____________	Velocity ____________	Acceleration ____________
Interval ____________	Velocity ____________	Acceleration ____________
Interval ____________	Velocity ____________	Acceleration ____________

The pendulum is slowing down when the "speed function" is decreasing. Activate (plot) the "speed function" again. Over what time intervals is the pendulum slowing down?

____.

Deactivate the speed function and activate (plot) the velocity and acceleration functions one more time. Rewrite the intervals above and tell whether the velocity and acceleration functions are positive or negative at these times.

Interval ________ Velocity ________ Acceleration ________
Interval ________ Velocity ________ Acceleration ________
Interval ________ Velocity ________ Acceleration ________
Interval ________ Velocity ________ Acceleration ________

What are the patterns that you see. Include these patterns/conclusions in your report.

____.
____.
____.
____.
____.

Let's do one more thing. According to the Newton's second law of physics, the period of a pendulum can be found from the following formula, which you will derive in Part B of this report:

$$\text{Period} = 2\pi\sqrt{\frac{L}{980}}$$

where L is the length of the pendulum in centimeters.

1. Using this formula what is the period of your pendulum? ____

2. Was your timing good? ____

Be sure to save your plots and your collected data for use in the report that you must write.

Student Task #1

Formally write up a summary of what was accomplished in this section. Include your graphs and explanations. Write it up so that a reader who has not performed the experiment nor seen these activity sheets could follow what was done and why. Do *not* turn in these papers, these are to be used to help organize your report.

Part B

Many motions of an object (e.g. the orbit of Earth about the sun, the trajectory of a baseball, the deformation of a spring) can be predicted by using an equation based on Newton's second law. Newton's second law states that *Force equals mass times acceleration.* This is a fundamental physical premise, and the exact form of the equation representing Newton's second law depends on the particular problem (e.g. the equation describing the Earth's orbit does not look like the equation which describes the deformation of a spring). It is our task to derive the equation for $d(t)$ from Newton's second law using reasonable assumptions and then to compare this equation with the equation you obtained in Part A, which was based purely on experimental data. If the assumptions we make are reasonable for our pendulum, then these two equations should be very similar.

I. Analytical Model of the Pendulum Motion

Shown in Figure 2 is a pendulum consisting of a ball and string with a chosen coordinate system. Let $x(t)$ be the x-coordinate of the ball and $y(t)$ be the y-coordinate of the ball. In order to derive the equation of motion we must invoke Newton's second law which states that the sum of all the forces acting in the x-direction is equal to the mass times the acceleration in the x-direction, and likewise, the sum of all the forces acting in the y-direction is equal to the mass times the acceleration in the y-direction. Mathematically we write:

$$\sum F_x = m\frac{d^2x}{dt^2} \tag{2}$$

$$\sum F_y = m\frac{d^2y}{dt^2}. \tag{3}$$

The forces in the above equations are shown in what is called a *free-body diagram* for the ball in Figure 3a. In the free-body diagram, the two forces that the ball feels are $\boldsymbol{T}$, the *tension* in the string which holds the ball up and $\boldsymbol{W}$, the *weight* (mass times gravity, mg) which tries to pull the ball down. Note that both of these forces have a magnitude (how

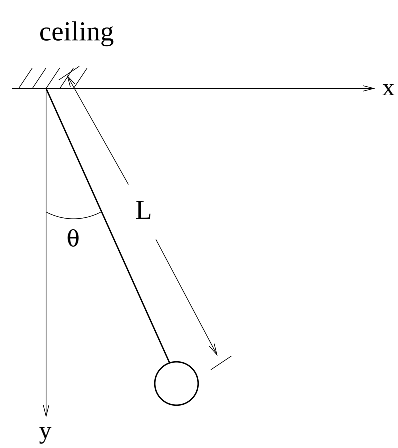

Figure 2: Simple Pendulum

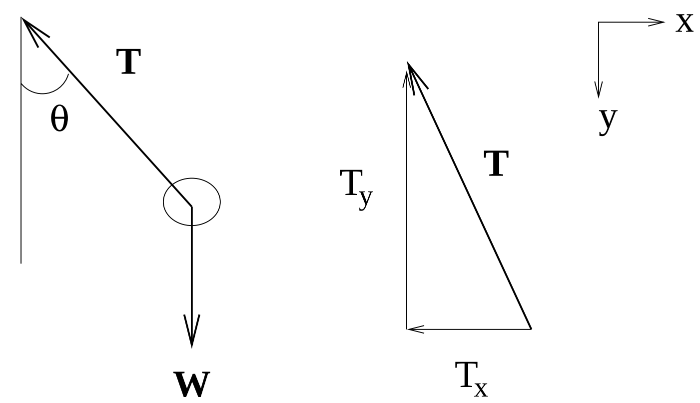

Figure 3: (a) Free Body Diagram of Ball, (b) Resolution of Tension into x and y Components

much tension the string feels, or what the weight of the ball is) and a direction (tension always points in the direction of the string, and weight always points down).

Figure 3b shows that we can represent tension as two components, one acting in the x-direction and one acting in the y-direction. The different components have magnitude (or length) necessary to make the lengths of $\boldsymbol{T}$, T_x, and T_y form a right triangle. To get a feel for this, answer the following questions. Let T be the magnitude of the tension.

1. What are the components of tension when the pendulum is straight up and down ($\theta = 0$)?
   $T_x =$ ____________________T $\qquad$ $T_y =$ ____________________T

2. What are the components of tension when the pendulum is at ($\theta = \pi/6$)?
   $T_x =$ ____________________T $\qquad$ $T_y =$ ____________________T

Using a free-body diagram (Figure 3a) as an aid, the equations representing Newton's second law, equations (2) and (3), may be written more specifically for the pendulum as:

$$m\frac{d^2x}{dt^2} = -T\sin(\theta) \tag{4}$$
$$m\frac{d^2y}{dt^2} = -T\cos(\theta) + mg. \tag{5}$$

Explain in words where each term, and its associated sign, comes from.

____.
____.
____.
____.
____.

Equations (4) and (5) contain the unknown functions x, y, T, and θ which are all functions of time, t (it is assumed that the mass, m, and the magnitude of the gravitational force,

g, are known). So right now we have 2 equations, (4) and (5), and 4 unknown functions. In order to be able to solve for the unknowns, we need the same number of equations as unknowns. So we need to find two more equations. Fortunately, we know something about the geometry of the problem, and this gives us the following:

$$x = L\sin(\theta) \qquad \text{or} \qquad x(t) = L\sin(\theta(t)) \tag{6}$$
$$y = L\cos(\theta) \qquad \text{or} \qquad y(t) = L\cos(\theta(t)), \tag{7}$$

where L is the length of the pendulum, which we assume to be a constant.

So we now have 4 equations and 4 unknowns and we should be able to solve for our unknowns. To get this down to something manageable, we would like to use substitution to eliminate 3 of the unknowns and get down to one equation and one unknown:

Student Task #2

Derive the *equation of motion*:

$$mL\frac{d^2\theta}{dt^2} + mg\sin\theta = 0. \tag{8}$$

Begin with equations (2) and (3), motivate each step and explain in words where terms come from. Hints follow.

Hints:

1. Begin with equations (4) and (5). Multiply equation (4) by $\cos\theta$; multiply equation (5) by $-\sin\theta$; then add these two equations together. What have we accomplished? (Which variable/function did we eliminate?).
2. Next, using equations (6) and (7), find $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$. (Don't forget, the chain rule is needed because θ is a function of t!).
3. Substitute your equations for $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$ from hint 2 into your resulting equation from hint 1. Perform some algebra and trig on this equation to show that it is equivalent to equation (8).

The equation of motion may be rewritten as

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0 \qquad \text{(divide equation (8) by } mL\text{).}$$

This equation is a *nonlinear differential equation* and is difficult to solve. One way to make the equation easier to solve is to approximate $\sin\theta$ by θ. To see why this approximation can be made, graph $y = \sin x$ and $y = x$. For what values of x are the graphs close to one another? ____.

Using this approximation ($\sin\theta = \theta$ for small values of θ) the above nonlinear equation can now be written as:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0. \tag{9}$$

You will learn a rigorous method for finding a solution to this sort of differential equation in class, but for now we will give you the solution and it will be your task to show that this is the solution. First, there are lots of solutions to (9), so to get one of the solutions we need to provide the *initial conditions*, or the state of the pendulum at time $t = 0$. Suppose we start the pendulum from rest (i.e. not moving: $\frac{d\theta}{dt}(0) = 0$) at a given initial angle ($\theta(0) = \theta_0$). Then formally we say that we must find the solution to the following differential equation subject to the initial conditions:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0, \tag{10}$$

$$\theta(0) = \theta_0 \qquad \frac{d\theta}{dt}(0) = 0. \tag{11}$$

Student Task #3
Show that the function

$$\theta(t) = \theta_0 \cos\left(\sqrt{\frac{g}{L}}\, t\right) \tag{12}$$

is a solution to differential equation (10) and that it also satisfies the initial conditions in equation (11).

Student Task #4
Show that the period of the motion given in student task #3 above is

$$\text{Period} = 2\pi\sqrt{\frac{L}{g}}.$$

(Note: This only takes one or two steps!)

One way of obtaining a solution to the differential equation (10) with initial conditions (11) is to 'guess' the form of the solution, and then determine the coefficients in the guessed form:

Student Task #5
'Guess' that the form of the solution to the differential equation (10) with initial conditions (11) is:

$$\theta(t) = A\cos(\alpha t) + B\sin(\beta t). \tag{13}$$

By applying this equation to the initial conditions (equation (11)) and the differential equation (equation (10)) show that the result is the solution given in Student Task #3. That is, show that

1. A in equation (13) must equal θ_0 (use equation (11)),
2. B in equation (13) must equal 0 (use equation (11)), and
3. α in equation (13) must equal $\sqrt{\frac{g}{L}}$ (use equation (10)).

II. Comparison between Empirical Model and the Physics-Based Model

We now have a physics-based model, equation (12), and an empirical model (model based on experiments), equation (1) of Part A. However the physics-based model is in terms of the angle θ while the empirical model is in terms of the distance from the wall, $d(t)$. So we will need to do some work before we can compare the two. Using the geometry shown in Figure 4, we will re-write the physics-based model in terms of $d(t)$.

Student Task #6

1. Using Figure 4 and the small angle approximation $\sin\theta \approx \theta$ show that

$$d(t) = D - L\theta.$$

2. Using the physics-based model for $\theta(t)$, (12), determine that

$$d(t) = D - L\theta_0 \cos\left(\sqrt{\frac{g}{L}}\, t\right).$$

3. Compare the parameters in this equation, which is based on physics and geometry, with that determined in Part A, equation (1).

Open Discussion: In engineering problems, engineers must design things (engines, buildings, bridges, computer chips) at a minimal cost. Sometimes it is too expensive to perform real-time experiments on an object (will a particular building withstand an earthquake of magnitude 6?), so that the engineer must make design specifications based purely on a physics-based model with reasonable assumptions. Safety factors are then built in to account for these assumptions. Other problems may allow engineers to perform experiments

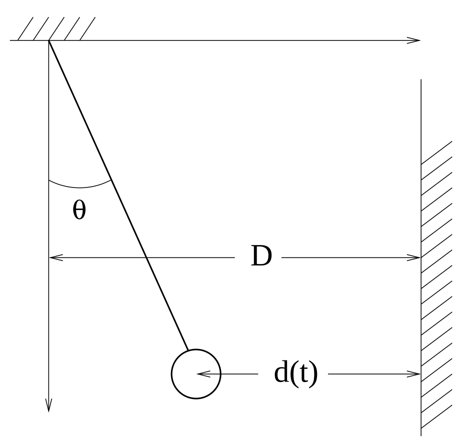

Figure 4: Geometry for Simple Pendulum

which complement a physics-based model (e.g. using a wind tunnel to help design an airplane wing), and other problems (e.g. when dealing with soil which may vary markedly over short ranges) may require a model to be developed purely on experiments since trying to develop a physics-based model becomes too complicated to solve, even with today's computational power.