REAL ANALYSIS   Bennethum   Homework 11

1. (10 pts) Chapter 5 problem 1 of Royden. Also - find $\frac{d}{dx}f(x)$ for $x \neq 0$ (use Calc 1 techniques). Then take the limit as x goes to 0. What can you conclude from this?

2. (20 pts) Suppose that $f : [a, b]$ is a non-decreasing, bounded function.
   (a) (7 pts) Prove that f has a left and right-hand limit at every point in $[a, b]$. (Hint: Show that $\lim_{x \to c^-} f(x) = \sup\{f(x) | a \leq x < c\}$).
   (b) (7 pts) Prove that the set of points in $[a, b]$ where f is discontinuous is at most countably infinite (hence f is Riemann integrable). (Hint: let $\delta(x) := \lim_{x \to c^+} f(x) - \lim_{x \to c^-} f(x)$, and consider the sets $A_n = \{x | \delta(x) > 1/n\}$.)
   (c) (6 pts) Prove that

$$\lim_{n \to \infty} n \int_a^{a+1/n} f(x)\, dx = f(a^+).$$

3. (10 pts) Suppose that $f : [a, b] \to I\!\!R$ is an increasing function. We proved in class that $\{x | D^+ f > D_- f\}$ is a set of measure zero. In this problem you will complete the proof that all the derivates of f are equal almost everywhere. To do this, you need only prove that $\{x | D_+ f < D^- f\}$ is a set of measure zero. (The proof of this is very similar to the case in class). After proving this we have:

$$D_- f \leq D^- f \leq D_+ f \leq D^+ f \leq D_- f$$

almost everywhere, i.e. that the derivates are all the same on $[a, b]$ a.e.

4. (5 pts) Let $UC[a, b]$ be the set of all uniformly continuous functions over the interval $[a, b]$. Is $UC[a, b] \subseteq AC[a, b]$ or $AC[a, b] \subseteq UC[a, b]$ or neither? Prove.

5. (10 pts)
   (a) Suppose $f \epsilon L^1(I\!\!R)$ and define

$$F(x) = \int_x^{x+1} f(t) dt.$$

   Prove that F vanishes at infinity.
   (b) Suppose that f is AC on $[-N, N]$ for each integer N, and $f' \epsilon L^1(I\!\!R)$ ($\int_{I\!\!R} |f'| < \infty$). Compute

$$\lim_{x \to \pm\infty} |f(x + 1) - f(x)|.$$

   Is it true that f vanishes at infinity?