

These are practice problems for test 2. For this exam, one side of an 8.5x11" sheet of paper will be allowed for notes. No technology of any kind will be allowed. The topics covered on this exam include material covered in Chapters 3 and 4 of Kohler and Johnson. Answers are given on the last page.

1. Prove the principle of superposition: If y_1 and y_2 are solutions to

$$y'' + p(t)y' + q(t)y = 0$$

then $y = c_1y_1 + c_2y_2$ is also a solution.

2. The principle of superposition applies to (circle one):
- A) Linear second-order homogeneous differential equations
 - B) Nonlinear second-order homogeneous differential equations
 - C) Linear second-order non-homogeneous differential equations
 - D) Nonlinear second-order non-homogeneous differential equations
3. Consider the following ODE:

$$y'' + 2y' + y = e^{3t}$$

- (a) Give a physical interpretation in terms of a spring-mass-dashpot system.
 - (b) Find the general solution.
 - (c) Is the *homogeneous* equation over-damped, under-damped or critically damped?
4. Find the general solution to

$$\mathbf{y}' = \mathbf{A}\mathbf{y}$$

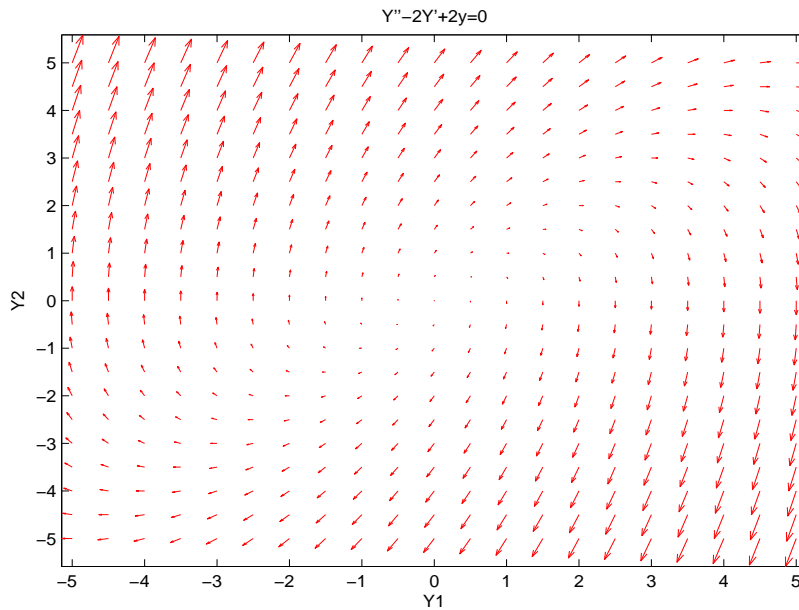
where

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$$

5. Consider the differential equation

$$y'' - 2y' + 2y = 0 \quad y(0) = 0, \quad y'(0) = 2$$

- Find two fundamental solutions and prove they are fundamental solutions.
- Write down the general solution.
- Determine the particular solution.
- Write the equation as a system of two first-order linear differential equations, $\mathbf{y}' = \mathbf{A}\mathbf{y}$. What are the initial conditions for this system?
- Find the eigenvalues and eigenvectors of \mathbf{A} .
- Write the general solution.
- Determine the particular solution. Is it consistent with your answer in (c)?
- The direction field is given below. On the phase plane plot your particular solution for the given initial condition. Is your solution consistent with the direction field?



1) y_1 and y_2 are solutions mean:

$$y_1'' + p(t)y_1' + q(t)y_1 = 0 \quad y_2'' + p(t)y_2' + q(t)y_2 = 0.$$

We need to show that $c_1y_1 + c_2y_2$ is also a solution. So we plug this into the left-hand side of the equation and show that it is equal to zero:

$$\begin{aligned} (c_1y_1 + c_2y_2)'' + p(t)(c_1y_1 + c_2y_2)' + q(t)(c_1y_1 + c_2y_2) \\ = c_1y_1'' + c_2y_2'' + p(t)c_1y_1' + p(t)c_2y_2' + q(t)c_1y_1 + q(t)c_2y_2 \\ = c_1[y_1'' + p(t)y_1' + q(t)y_1] + c_2[y_2'' + p(t)y_2' + q(t)y_2] \\ = c_1(0) + c_2(0) = 0. \end{aligned}$$

2) A The proof for problem 1 does not work for any of the other cases.

3a) (may want to draw picture, in metric units): a mass of 1 Kg hangs from a system with a dashpot of 2 Newtons-sec/meter and a spring of 1 Newton/m. An external force of e^{3t} acts on the system in the vertical direction.

3b) $y = c_1e^{-t} + c_2te^{-t} + \frac{1}{16}e^{3t}$.

3c) The homogeneous solution is $y = c_1e^{-t} + c_2te^{-t}$ indicating that it's not underdamped (doesn't oscillate), and is critically damped because the discriminant is zero.

4)

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1e^{2t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2e^{-t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

5a) Fundamental solution set: $\{e^t \cos t, e^t \sin t\}$. The Wronskian evaluated at $t = 0$: $W(0) = 1 \neq 0$.

5b) $y = Ae^t \cos t + Be^t \sin t$.

5c) $y = 2e^t \sin t$.

5d)

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

5e) $\lambda_1 = 1 + i, \quad \lambda_2 = 1 - i, \quad v_1 = [1, 1 + i]^T \quad v_2 = [1, 1 - i]^T$.

5f)

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1e^t \begin{bmatrix} \cos t \\ \cos t + \sin t \end{bmatrix} + c_2e^t \begin{bmatrix} \sin t \\ \cos t + \sin t \end{bmatrix}$$

5g)

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = 2e^t \begin{bmatrix} \sin t \\ \cos t + \sin t \end{bmatrix}$$

so $y = y_1 = 2e^t \sin t$ which is the same as what was obtained in (c), so we have consistency :). Also note that y_2 is the derivative of y_1 .

5h) Starts at $(0, 2)$ and spirals out clockwise.