Math 3200 Mini-Projects

(modified from Bill Briggs 2004)

This collection of assorted mini-projects is supported by the material that we will study this semester. You must complete two (2) mini-projects during the semester. The first mini-project is due no later than **March 4** and must be selected from projects 1–4. The second mini-project is due no later than **May 7** and must be selected from projects 5–10. Do not procrastinate; there will be no extensions! Each project will determine 7% of your grade. You may collaborate on the projects, but the your final write-up *must be entirely your own work*.

The solutions to the mini-projects must be neat, legible, and written with perfect spelling and grammar. It would be best to use a word processor or LaTeX to write at least the text portions of every assignment. If you do write by hand, it must be effortless to read. A typical solution should consist of

- a statement or summary of the problem;
- a description of the overall approach to the problem;
- a well-justified solution;
- a presentation of results including relevant tables and graphs; and
- a discussion that includes an interpretation of the solution, an assessment of accuracy, and/or other interesting observations.

Do not base your choice of projects on their apparent length! The projects are designed to require *roughly* the same amount of time and effort if you are prepared. Some are quite applied and others are more theoretical in nature. The projects are listed more or less in the order in which we will study the relevant material in class. You should be able to get started almost immediately. If you want to work on a project that requires material we have not yet covered in class, I'll be glad to help you get started. I am always available for questions and consultation on the projects. Most of all, have fun with them!

1. **Evaporating Reservoirs**. Imagine a large water reservoir that loses water due to evaporation. In all that follows, we will let $h(t)$, $S(t)$ and $V(t)$ denote the depth, the surface area, and the volume of the water in the reservoir, respectively, at time $t \geq 0$. We will always assume that the rate of change of the water volume is proportional to the surface area of the exposed water in the reservoir; that is, $V'(t) = -\alpha S(t)$, where $\alpha = 0.05$ meters/day (notice the minus sign).

   (a) First (warm-up) consider a reservoir that has the shape of rectangular prism (a box or parallelepiped) with a constant horizontal cross-sectional area of 200 square meters and a depth of 10 meters (see left figure below).

      i. Verify that the units of α are consistent.

      ii. Assuming that the reservoir is filled at $t = 0$, what is the initial volume of the water?

      iii. Noting that in this case the surface area of the reservoir is constant, solve the ODE that governs the change in water volume. Use the initial condition to express the volume as a function of time.

      iv. At what time (after how many days) will the reservoir be empty, assuming the evaporation rate α remains constant?

   (b) Now (the real problem) assume that the reservoir is shaped like an inverted frustrum of a pyramid: an upside-down square-based pyramid, the top of which has been sliced off (see right figure below). The horizontal cross-sections of the reservoir are always squares which decrease in area from 225 square meters at the top of the reservoir to 100 square meters at the base. The depth of the reservoir is 10 meters.

      i. The volume of a pyramid is given by $V = \frac{1}{3}Ah$ where A is the area of the (square) base and h is the height. Verify that the volume of water in the reservoir when it is full is $V_0 = 1583.33$ cubic meters.

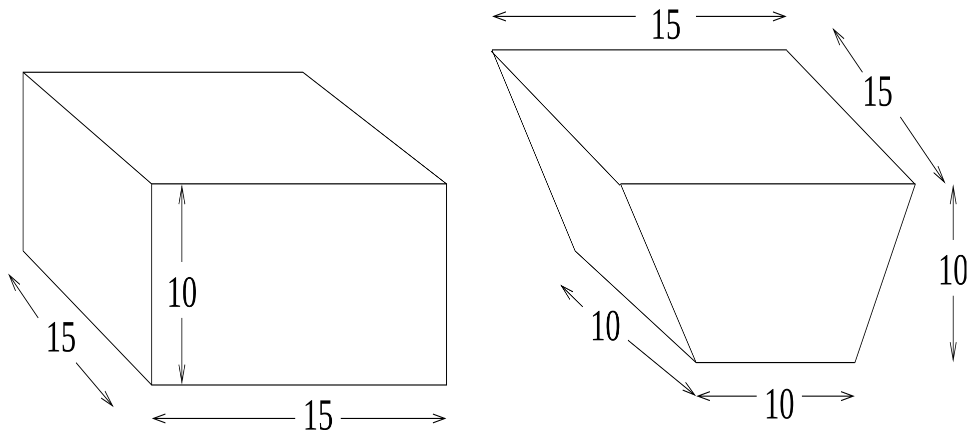

Figure 1: *Box reservoir (left) and pyramid reservoir (right) for Project 1.*

ii. Letting h denote the water depth in the reservoir, verify that the surface of the water in the reservoir is a square whose sides have length given by $\ell(h) = 10 + h/2$. (Check that $\ell(0) = 10$ and $\ell(10) = 15$.)

iii. Show that the surface area of the water is $S(h) = (\ell(h))^2$. (Check that $S(0) = 100$ and $S(10) = 225$.)

iv. Show that the volume of water in the reservoir is given by $V(h) = \frac{1}{3}S(h)(20+h) - \overline{v}$, where $\overline{v} = 666.67$ cubic meters. (Check that $V(0) = 0$ and $V(10) = V_0$.) A derivation is required.

v. Important step: In order to use the ODE $V'(t) = -\alpha S(t)$, we must relate the surface area directly to the volume. Show that $S = \frac{1}{4}(12(V + \overline{v}))^{2/3}$.

vi. Show that the governing ODE for the volume is $V'(t) = -a(V + \overline{v})^{2/3}$ where $a = 3\alpha/\sqrt[3]{12}$.

vii. Solve this ODE (using the initial condition $V(0) = V_0$) and graph the solution.

viii. At what time (after how many days) will the reservoir be empty, assuming the evaporation rate α remains constant?

2. **Free Fall and Terminal Velocity**. An object in free fall in a gravitational field is governed by the ODE

$$m\frac{dv}{dt} = mg + F_e,$$

where m is the mass of the object, $g = 9.8$ meters/sec^2 is the acceleration of gravity, $v(t)$ is the velocity of the object t seconds after it is released, and F_e denotes external forces acting on the object. In all that follows, assume that $v(0) = 0$ and that the positive direction for velocity and position is *downward*, in the same direction as g.

(a) If there are no external forces acting on the object, then its velocity increases without bound (until the object collides with something). This is unrealistic for motion in the earth's atmosphere, because air resistance is a significant effect. Therefore, assume that air resistance is present and is described by $F_e = -kv$ (k is a constant and the minus sign indicates that the air resistance opposes the motion). Solve the initial value problem (include all steps) and show that the solution of this ODE is

$$v(t) = \frac{mg}{k}(1 - e^{-kt/m}).$$

(b) What is the terminal velocity $v_T = \lim_{t\to\infty} v(t)$ of a 100-kilogram object (a small linebacker or a large flower pot) subject to air resistance described by $k = 5$ kg/sec?

(c) Find the function that describes the position $x(t)$ of the object for all $t \geq 0$ assuming that $x = 0$ corresponds to the position at which the object is dropped.

(d) Make rough sketches of $v(t)$ and $x(t)$.

(e) Now assume that $F_e = -kv^2$ (this is generally a more accurate way to model air resistance). Solve the resulting ODE for the velocity of the object.

(f) With values of $m = 100$ kg and $k = .1$ kg/meter, what is the terminal velocity of the object? (Notice that the k's that appear in the two models are different.)

(g) Find the function that describes the position $x(t)$ of the object for all $t \geq 0$ assuming that $x = 0$ corresponds to the position at which the object is dropped.

(h) Make rough sketches of $v(t)$ and $x(t)$.

(i) Compare and contrast the two models for air resistance.

3. **Periodic Drug Doses**. Most drugs are eliminated from the body according to a strict exponential decay law.

   (a) Warm-up. The drug valium has a half-life in the blood of 36 hours (a population average). Assume that a 50-milligram dose of valium is taken at time $t = 0$. Let $m(t)$ be the amount of drug in the blood in milligrams t hours after the dose. Plot the function $m(t)$ as it varies with time. After how many hours, will the amount of drug reach 10% of its initial value? After how many hours, will the amount of drug reach 1% of its initial value?

   (b) Real problem. Now imagine that a drug (such as aspirin or an antibiotic) with a half-life of 12 hours is taken regularly every 8 hours in doses of 50 milligrams. Assume that the first dose is taken at time $t = 0$. Let m_k denote the amount of drug in the blood immediately prior to the kth dose and let M_k denote the amount of drug in the blood immediately after the kth dose; thus, $m_1 = 0$ and $M_1 = 50$.

      i. Find m_2, the amount of drug in the blood at $t = 8$ hours just *prior* to the second dose.
      ii. Find M_2, the amount of drug in the blood at $t = 8$ hours just *after* to the second dose.
      iii. Find m_3, the amount of drug in the blood at $t = 16$ hours just *prior* to the third dose.
      iv. Find M_3, the amount of drug in the blood at $t = 16$ hours just *after* to the third dose.
      v. Now generalize: Find m_k, the amount of drug in the blood just *prior* to the kth dose, where $k = 1, 2, 3, \ldots$.
      vi. Find M_k, the amount of drug in the blood just *after* to the kth dose. Sketch a rough graph showing the amount of drug in the blood as it varies in time.
      vii. What can you say about the long-term amount of the drug in the blood? Does it continue to increase without bound or does it approach a steady-state level? If you argue for the latter choice, find the steady-state value of the amount of drug. Justify your conclusion carefully.
      viii. Apply the periodic doses problem to solve the following problem *quickly*: A fish hatchery harvests 1/3 of its current fish population at the end of each year, and then immediately replenishes the population with 500 new fish. Assuming no deaths and an initial fish population of 500 fish, what is the steady state population in the hatchery?

4. **Numerical Solutions of ODEs** (Computer programming required). The goal of this project is to write a computer program that uses Euler's method to approximate solutions to ODEs and systems of two ODEs. Descriptions of Euler's method can be found in many ODE and numerical analysis texts, including our current text.

   (a) Write a program that uses Euler's method to approximate the solution to the general first-order initial value problem $y'(t) = f(t, y), \quad y(a) = y_0$, where f is a user-supplied function. Test the program on the initial value problem

   $$y'(t) + \frac{1}{t}y(t) = t^p, \quad y(1) = 10 \quad \text{for} \quad 1 \leq t \leq 10.$$

   Compare your Euler solutions to the exact solution in the cases $p = -1$ and $p = 0$. Discuss how the error in your Euler solutions varies with the step size.

(b) Modify your program (slightly) to handle two coupled ODEs of the form $y'(t) = f(t, y, z)$ and $z'(t) = g(t, y, z)$, where f and g are user-supplied functions. Use the program to solve the pendulum problem $\theta''(t) + \sin\theta = 0$, which can be written as the two first order ODEs

$$\begin{aligned} y'(t) &= z \\ z'(t) &= -\sin y. \end{aligned}$$

Use the initial values $\theta(0) = y(0) = A$ and $\theta'(0) = z(0) = 0$, where the initial amplitude of the oscillation A can be varied. Display your Euler solutions graphically and experiment with the dependence of the solutions on A and the step size.

5. **World Record Mile Times**. The table below shows the *difference* between the world records in the mile for women and men in five decades. Is there reason to believe that sometime in the future the two records will be equal? We will use a technique called *linear regression* or *linear least squares* to address the question.

Year	**Difference in World Records (sec.)**
1960	54
1970	46
1980	40
1990	34
2000	26

(a) Let's first work out the linear least squares approach in general. Suppose that you are given n data pairs (t_i, y_i) for $i = 1, 2, \ldots, n$. Clearly, in general, a single straight line cannot pass through more that two points. So the goal is to find the line $f(t) = at + b$ that gives the "best fit" to these data points. There are many ways to define best fit, but for least squares, the idea is to minimize a measure of the total *vertical distance* between the data points and the best fit line; that is, we will find the values of a and b that minimize the quantity

$$E = \sum_{i=1}^{n} (f(t_i) - y_i)^2 = \sum_{i=1}^{n} ((at_i + b) - y_i)^2.$$

Recall from Calculus III that (necessary) conditions for minimizing E can be found by taking the derivatives $\frac{\partial E}{\partial a}$ and $\frac{\partial E}{\partial b}$, setting them equal to zero, and solving for a and b. Carry out this calculation (carefully) to find the coefficients a and b for the least squares line.

(b) Let $x = (a, b)^T$, let A be the $n \times 2$ matrix whose ith row is $(t_i, 1)$, and let $y = (y_1, \ldots, y_n)^T$. Show that the least squares problem amounts to solving the overdetermined system of equations $Ax = y$.

(c) Apply part (a) to the world record time data to find the least squares line. Plot the data points and the least squares line.

(d) Based on the least squares line, can you conclude that the world record for women and men will be equal in the future? When will this occur? Comment on the validity of this prediction.

(e) Referring to parts (b) and (c), show the the solution you found, $x = (a, b)^T$, satisfies the system of equations $A^T A x = A^T y$.

6. **Period of the Pendulum**. The full (nonlinear) equation of motion for an undamped pendulum is

$$\theta''(t) + \omega^2 \sin\theta(t) = 0,$$

where $\omega^2 = g/\ell$, g is the acceleration of gravity, ℓ is the length of the pendulum, and θ is the angular displacement of the pendulum measured in radians. (The more familiar linear equation results by assuming that $\theta << 1$ (small amplitudes) and that $\sin\theta \approx \theta$.) An explicit solution of this nonlinear ODE cannot be found in terms of familiar functions. However, it is possible to determine the period of the nonlinear pendulum. Assume the initial conditions $\theta(0) = \alpha$ and $\theta'(0) = 0$. We will consider a quarter period corresponding to θ decreasing from $\theta = \alpha$ to $\theta = 0$.

(a) What is the period of the linear pendulum governed by $\theta'' + \omega^2\theta = 0$?

(b) Now let's work on the full nonlinear ODE. Multiply both sides of the ODE by θ', use the chain rule carefully, and apply the initial conditions to show that

$$\theta'(t) = \pm\omega\sqrt{2(\cos\theta(t) - \cos\alpha)}.$$

(Note that $((\theta')^2)' = 2\theta'\theta''$.)

(c) Do you choose the plus or minus branch of the square root? During the quarter period we are considering, is $\theta' < 0$ or is $\theta' > 0$?

(d) Now separate variables and write

$$dt = -\frac{1}{\omega}\frac{d\theta}{\sqrt{2(\cos\theta - \cos\alpha)}}.$$

(e) Show how the identity $\cos x = 1 - 2\sin^2(x/2)$ leads to

$$dt = -\frac{1}{2\omega}\frac{d\theta}{\sqrt{\sin^2(\alpha/2) - \sin^2(\theta/2)}}.$$

(f) The function on the right is still difficult to integrate, so we define a new variable ϕ by $\sin(\theta/2) = \sin(\alpha/2)\sin\phi$ (recall that α is known). After changing variables you should have

$$dt = -\frac{1}{\omega}\frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}},$$

where $k = \sin(\alpha/2)$.

(g) Now we can integrate. Notice that when the original variable θ varies from $\theta = \alpha$ to $\theta = 0$, the new variable ϕ varies from $\phi = \pi/2$ to $\phi = 0$. Letting T be the full period and integrating over a quarter period, show that

$$\frac{T}{4} = \frac{1}{\omega}\int_0^{\pi/2}\frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}}.$$

(h) This last integral is called an **elliptic integral of the first kind** and is denoted as $F(k, \pi/2)$. Therefore, we have found that the period of the pendulum is $T = (4/\omega)F(k, \pi/2)$.

(i) What is the period in the limiting case $k = 0$ which corresponds to $\alpha = 0$.

(j) Use a table of elliptic integrals to find the period of a pendulum with $\ell = 2$ meters with initial displacements of $\alpha = \pi/12, \pi/6, \pi/3, \pi/2$.

(k) Briefly comment on how the behavior of the linear and nonlinear pendula differ.

7. **Population Genetics**.

You may recall from a biology course that the simplest genetic traits are transmitted by a single gene that has one of two forms or *alleles*. For example, a gene for eye color may be **A** for the dominant allele (brown eyes) or **a** for the recessive allele (blue eyes). With one gene coming from each parent, an offspring may have one of three *genotypes*: AA (brown eyes), Aa (brown eyes), or aa (blue eyes). Suppose that mating takes place randomly (the genes are thoroughly mixed) and that the three genotypes have different fitnesses (or survival probabilities) b, c, and d, respectively. We will let $p(t)$ represent the fraction of **A** alleles in the gene pool at time t. This means that $0 \leq p \leq 1$ and the fraction of **a** alleles is $1 - p$. The differential equation that governs how p varies in time is

$$p'(t) = p(1-p)((b - 2c + d)p + c - d).$$

(a) What are the equilibrium points for this ODE?

(b) Draw the direction field for the case that $b = 0.2, c = 0.4$ and $d = 0.6$ and determine the stability of the equilibrium points. Determine $\lim_{t\to\infty} p(t)$.

(c) Draw the direction field for the case that $d = 0.2, c = 0.4$ and $b = 0.6$ and determine the stability of the equilibrium points. Determine $\lim_{t\to\infty} p(t)$.

(d) Draw the direction field for the case that $b = 0.2, d = 0.4$ and $c = 0.6$ and determine the stability of the equilibrium points. Determine $\lim_{t\to\infty} p(t)$.

(e) Draw the direction field for the case that $c = 0.2, b = 0.4$ and $d = 0.6$ and determine the stability of the equilibrium points. Determine $\lim_{t\to\infty} p(t)$.

(f) Find the general conditions on b, c, and d for *polymorphism*, the coexistence of both genes in the steady state (that is, $p \neq 0$ and $p \neq 1$ in the steady state).

8. **Epidemics and Markov Chains**. Consider a population in the midst of an epidemic. All individuals in the population can be classified as *healthy*, *infected*, or *deceased*. On average, each month, $\frac{3}{10}$ of those who are healthy become infected, and $\frac{1}{4}$ of those who are infected die. Assume that no healthy person dies without first becoming infected. Let H_k, I_k, and D_k denote the fraction of the population that is healthy, infected, and dead, respectively, at the end of the kth month. Let v_k be the column vector $(H_k, I_k, D_k)^T$. Assume the initial state v_0 is given.

   (a) Write the system of equations that relates v_{k+1} to v_k. Determine the transition matrix A such that $v_{k+1} = Av_k$ for $k = 0, 1, 2, \ldots$.

   (b) Prove that $H_k + I_k + D_k = 1$ for all k.

   (c) Find the eigenvalues and eigenvectors of A and use them to determine a general formula for v_k, for all $k = 0, 1, 2, \ldots$.

   (d) Given the solution in part (c), determine the steady state solution. What is the ultimate state of the population.

   (e) Comment on the validity of this model.

9. **A Predator-Prey Model**. The Lotka-Volterra model for describing the interaction between a predator and a prey was formulated in the early 20th century. It has been shown to be fairly accurate when applied to many natural systems (lynx-rabbits, sharks-fish). Let $F(t)$ and $R(t)$ denote the population of a predator and a prey species (think Fox and Rabbit) at time $t \geq 0$, measured in hundred of individuals. Consider the predator-prey equations

$$\begin{aligned} F'(t) &= -6F + 2FR \\ R'(t) &= 12R - 3FR. \end{aligned}$$

   (a) First give a brief interpretation of the equations. What is the effect of an increase in the rabbit population on the existing rabbit and fox populations? What is the effect of an increase in the fox population on the existing rabbit and fox populations?

   (b) For what fox and rabbit populations is the system at equilibrium?

   (c) Sketch the direction field for these equations in the phase plane. (You need to consider only $R > 0$ and $F > 0$. Why?) Choose a few different initial conditions and sketch the resulting trajectories in the phase plane.

   (d) Divide the predator ODE by the prey ODE (or vice-versa) to obtain a single ODE in R and F (with t no longer present). Solve this separable ODE to find an implicit representation for the trajectories.

   (e) **10 pts extra credit** Show that that the average populations are given by

$$\overline{R} = \frac{1}{T}\int_0^T R(t)\,dt = 3 \quad \text{and} \quad \overline{F} = \frac{1}{T}\int_0^T F(t)\,dt = 4.$$

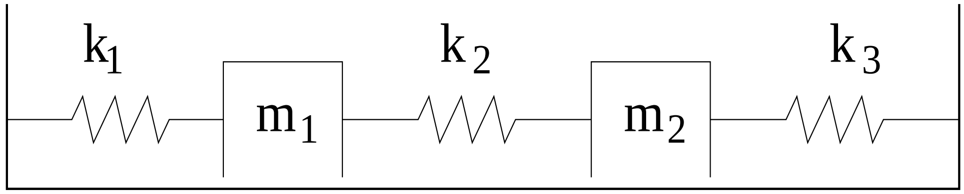

Figure 2: *Diagram of coupled oscillators for Project 13.*

10. **Coupled Oscillators**. Imagine two blocks of mass m_1 and m_2 on a frictionless horizontal surface attached to each other and to two walls by springs with spring constants k_1, k_2, and k_3 (see figure). Let $x(t)$ and $y(t)$ denote the displacement of the left and right block, respectively, from their equilibrium positions, with positive displacement to the right. At time $t = 0$, the left block is held at its equilibrium position ($x(0) = 0$) and the right block is displaced two units to the right of its equilibrium position ($y(0) = 2$). The governing equations for this system (assuming no friction and no external forcing) is

$$\begin{aligned} m_1 x''(t) &= -k_1 x + k_2(y - x) \\ m_2 y''(t) &= -k_2(y - x) - k_3 y \end{aligned}$$

(a) Briefly explain the meaning of each term of these equations.

(b) Write the system as a set of four linear ODEs in the variables x, x', y, y'. Determine the matrix for this system.

(c) Find the eigenvalues and eigenvectors of this system in the special case that $m_1 = m_2 = m$ and $k_1 = k_2 = k_3 = k$.

(d) Find the general solution of this system and implement the initial conditions to determine the arbitrary constants.

(e) Graph the displacement functions for the two blocks and comment on the behavior of the system.