Review of Material Needed for MATH 2421
CU-Denver Math Dept.

This is a packet of prerequisite material necessary for understanding material covered in Calculus 3. Many students take this course after having taken their previous course many years ago, at another institution where certain topics may have been omitted, or have not seen this material in a long time or just feel uncomfortable with it. Because understanding this material is so important to being successful in this course, we have put together this review packet. To evaluate how well you understand the material, you will be given a quiz over the material in this packet next week. The quiz will consist of material covered in this packet only, but no partial credit will be given.

In this packet you will find sample questions and a brief discussion of each topic. If you find the material in this pamphlet is not sufficient for you, it will be necessary for you to look in the appropriate sections of our textbook and understand the material on your own. Because this is considered pre-requisite material, it is ultimately your responsibility to learn it. The topics to be covered include: (1) Graphs and Equations of Lines and Conic Sections, (2) Polar Coordinates, (3) Parametric Equations, (4) Differentiation and Max/Min, and (5) Integration and Area Under the Curve.

1 Graphs and Equations of Conic Sections

Much of Calculus 3 involves developing your skill to visualize in 3 dimensions - with and without the help of a computer. To make this easier, it helps to be well acquainted with the graphs of some elementary 2-dimensional figures, such as lines, parabolas, circles, ellipses, and hyperbolas. Each of these figures are the graphs of an equation of the form: $ax^2 + by^2 + cx + dy + e = 0$. This material is covered in Appendix B of your text.

Exercise:

1. Match the graph with the equation.

(i) $\sqrt{2}\,x = \frac{y}{3} + 2$	A line
(ii) $x^2 - y = 2$	B circle
(iii) $x + 3y^2 + 7 = 0$	C parabola opening up
(iv) $\sin(5)x^2 = 8 - \sin(5)y^2 + 3x$	D hyperbola
(v) $6x^2 = 9 + 2y^2 + 3y$	E ellipse
(vi) $6x^2 = 9 - 2y^2 + 3y$	F parabola opening to the left

2. (Trick question - would never give this on a quiz). Graph $y^2 + 2xy + x^2 = 9$

Discussion:
Recall that the standard form of a **line** is $ax + by + c = 0$. This is characterized by both x and y being raised to the *first* power. To graph, it is simpler to put the line in slope-intercept form (assuming $b \neq 0$):

$$y = -\frac{a}{b}x - \frac{c}{b}$$

so that $-\frac{a}{b}$ is the slope and $-\frac{c}{b}$ is the y-intercept (when $x = 0$, $y = -\frac{c}{b}$). Two special cases include $a = 0$ so that the equation becomes $y = -\frac{c}{b}$, which means that for *any* x, y is $-\frac{c}{b}$, which is a horizontal line; and if $b = 0$ then we have $x = -\frac{c}{a}$ which means that for *any* y, $x = -\frac{c}{a}$ which means that we have a vertical line.

The **parabola** is characterized by the highest power of one of x, y being a 2, and the other variables being raised to the first power, e.g.

$$ax^2 + by + cx + d = 0 \qquad \text{or} \qquad ax + by^2 + cy + d = 0. \tag{1}$$

The signs of a and b affect which way it opens, and the other coefficients (c and d) affect the width of the parabola and the location of the vertex. The standard form (the form which makes sketching easiest) is

$$y - k = a(x - h)^2 \qquad \text{or} \tag{2}$$
$$x - h = b(y - k)^2 \tag{3}$$

where (2) is a parabola which opens up ($a > 0$) or down ($a < 0$), and (3) is a parabola which opens left ($b < 0$) or right ($b > 0$). In either case (h, k) is the vertex (note that by putting in $x = h$ and $y = k$ we get the equations are satisfied identically). To get the equation from the form of (1) to the standard form of (2) or (3) you may have to complete the square.

The **circle** is characterized by both x and y being squared with the same coefficient in front of x^2 and y^2 when they are on the same side of the equal sign:

$$ax^2 + ay^2 + bx + cy + d = 0. \tag{4}$$

So for example

$$-5x^2 + 8x - 9 - 5y^2 - 2y = 0 \qquad \text{and} \qquad 2x^2 = -2y^2 + 5$$

are both circles, since when everything is put on one side the coefficients of x^2 and y^2 are the same. The standard form (the form easiest to sketch from) of the circle is:

$$(x - h)^2 + (y - k)^2 = r^2 \tag{5}$$

where (h, k) is the center and r is the radius. Putting and equation of the form (4) into standard form involves dividing by a and completing the square. Thus from (4) we know that a affects the radius and all the coefficients affect the location of the center.

The **ellipse** is characterized by both x and y being squared with the coefficients of x^2 and y^2 being of the same sign, but possibly different values:

$$ax^2 + by^2 + cx + dy + e = 0, \tag{6}$$

where either both a and b are positive, or both are negative. The more a and b differ from each other, the more elongated the ellipse. The standard form of an ellipse oriented so that its major and minor axis are parallel to the x and y axis (not all ellipses may be so oriented) is

$$\frac{(x-h)^2}{A^2} + \frac{(y-k)^2}{B^2} = 1, \tag{7}$$

where (h, k) is the center of the ellipse, A is the distance from the center to the vertices in the direction parallel to the x-axis, and Y is the distance from the center to the vertices in the direction parallel to the y-axis. So for example

$$\frac{x^2}{9} + \frac{y^2}{4} = 1, \tag{8}$$

has center at $(0, 0)$ and vertices at $(\pm 3, 0)$ and $(0, \pm 2)$. Note that one can put in each of the 4 vertice points and equation (8) is satisfied identically. $(0, 0)$ does not satisfy equation (8), since the center does not lie on the curve of the ellipse.

The **hyperbola** is characterized by both x and y being squared with the coefficients of x^2 and y^2 being of *opposite* sign. The coefficients of x^2 and y^2 may be equal in magnitude or not:

$$ax^2 - by^2 + cx + dy + e = 0, \tag{9}$$

where both a and b are both positive or both negative. So for example

$$3x^2 = 3y^2 - 2y + 10 \qquad \text{and} \qquad 2x^2 - 5y^2 + 3x - 10y = 332 \tag{10}$$

are both hyperbolas since when everything is put on the same side of the equals sign, the coefficients of x^2 and y^2 are of opposite sign. The standard form (the form easiest to sketch from) of the hyperbola is:

$$\frac{(x-h)^2}{A^2} - \frac{(y-k)^2}{B^2} = 1 \qquad \text{or} \qquad -\frac{(x-h)^2}{A^2} + \frac{(y-k)^2}{B^2} = 1 \tag{11}$$

where (h, k) is the center and the location of the negative sign determines whether the hyperbola opens left-right or up-down (note that the right-hand-side is always $+1$). The values of A and B determine how narrow or wide the hyperbola is. We will not go into sketching the hyperbola here, but refer you to a pre-calculus text book.

Solution to Exercise:

1. Equation (i) is a line since both x and y are being raised to the first power. The square root of 2 is just a number, fixed, and is the coefficient of x.

   Equation (ii) is a parabola. Since it is the x which is squared, for every y value for which the equation makes sense $(2 - y \geq 0)$ there are two corresponding x-values. Thus the parabola must open up or down. Recalling that $y = x^2$ opens up, and that Equation (ii) has the same signs, $y = x^2 - 2$, this is a parabola which opens up.

   Equation (iii) is also a parabola since x is squared and y is not. For every x value for which the equation makes sense $(7 - x \geq 0)$ there are 2 y values, we know that the parabola opens left or right. Putting it approximately in standard form (just to check signs): $x = -3y^2 - 7$, we see that it opens to the left.

   Equation (iv) is a circle, since when we put everything on one side we have the coefficients of x^2 and y^2 are both $\sin(5)$. Recall that $\sin(x)$ is a function, but $\sin(5)$ is a constant. The presence of $3x$ indicates that the origin is not at $(0, 0)$, and the radius will not be $\sqrt{8}$.

   Equation (v) is a hyperbola since when putting everything on one side of the equals sign, the coefficients of x^2 and y^2 are opposite in sign.

   Equation (vi) is an ellipse, since when everything is on one side of the equals sign, the coefficients of x^2 and y^2 have the same sign but are not equal. If they were equal (say both were 6), then the equation would be a circle.

   Note that in all these examples, what matters is the coefficients of the *highest* order terms, e.g. if the highest order term of x is x^2 and the highest order term of y is one, then we have a parabola, and it is the coefficients of these terms which determines the direction it opens. Lower order terms generally affect location, direction, or narrowness/broadness.

2. At first glance this looks like a circle - *but* this has a cross term, xy which is not in standard form. In fact, this is the equation of 2 lines (!!). To see this, factor

$$
\begin{aligned}
&y^2 + 2xy + x^2 = (y + x)^2 = 9 \\
&y + x = \pm 3 \\
&y = \pm 3 - x
\end{aligned}
$$

Exercises:
Sketch:

1. $x^2 - 6x + y^2 = -5$
2. $x = 3y^2$
3. $9x^2 + 4y^2 = 25$

Hints to solutions of exercises:
1. This is a circle since the coefficients of both x^2 and y^2 are both 1. We need to put it into standard form by completing the square:

$$\begin{aligned}
&(x^2 - 6x + \quad) + y^2 = -5\\
&(x^2 - 6x + 9) + y^2 = -5 + 9\\
&(x - 3)^2 + y^2 = 4 \qquad \text{multiply this out to be sure it}'\text{s correct}
\end{aligned}$$

so the center is at $(3, 0)$ and the radius is 2. If after completing the square, the right-hand-side were negative, then this would mean *no* points could satisfy the equation, since the left-hand-side is always positive, and the graph should be blank. If this occurs it would probably be a good idea to check some points in the original equation to see if this makes sense or whether there was an algebraic error made somewhere along the way.

2. Since the y is squared and the x is not, this is a parabola which opens... after some thought... right. It's already in standard form, so the vertex is $(0, 0)$ (note that this satisfies the equation). Plugging in a couple of points should tell you whether the 3 causes the parabola to be wider or smaller (note that choosing points for y is easier than fixing points for x).

3. Since both x and y are squared and the coefficients are the same sign but unequal, this is an ellipse. Putting it in standard form gives:

$$\begin{aligned}
&\frac{9x^2}{25} + \frac{4y^2}{25} = 1\\
&\frac{x^2}{25/9} + \frac{y^2}{25/4} = 1\\
&\frac{x^2}{(5/3)^2} + \frac{y^2}{(5/2)^2} = 1
\end{aligned}$$

so that the center is the origin, $(0, 0)$, and the vertices are: $(\pm 5/3, 0)$ and $(0, \pm 5/2)$.

2 Polar Coordinates

For some problems, especially problems which have an axis of symmetry, polar coordinates, as oppose to cartesian or rectangular coordinates, is the natural system to set up a problem. In Calc 3, we will not only work with the equivalent of polar coordinates in 3 dimensions, but also introduce a third coordinate system, *spherical coordinates*. For this it is important that you are comfortable with polar coordinates and are familiar with the sin, cos, tan of the angles 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$, $\frac{2\pi}{3}$, $\frac{3\pi}{4}$, $\frac{5\pi}{6}$, etc. You will find this material in your text book in Appendix H.

Exercises:

1. Convert from cylindrical coordinates to rectangular coordinates: $(r, \theta) = (2, \frac{5\pi}{3})$.
2. Convert from rectangular coordinates to cylindrical coordinates: $(x, y) = (-\frac{\sqrt{3}}{2}, -\frac{3}{2})$.
3. Plot $r = 2$.
4. Plot $\theta = \frac{2\pi}{3}$.
5. Re-write the equation in polar coordinates $x = y$.

Discussion:
In rectangular coordinares, the coordinates are denoted by (x, y) where x represents the distance parallel to the x-axis and y represents the distance parallel to the y-axis away from the origin. In polar coordinates, the coordinates are denoted by (r, θ), where θ is the angle, measured counterclockwise, from the positive x-axis, and r is the (directed) distance away from the origin. r may be positive or negative, and θ is not restricted to being between 0 and 2π. Thus, although in rectangular coordinates each point may be represented uniquely by one pair of coordinates, in polar coordinates, there are multiple coordinates which may represent a single point. As an example, all of the following represent the same point:

$$\begin{array}{ll} (0,3) & \text{in rectangular coordinates} \\ (3,\pi/2),\ (-3,3\pi/2),\ (-3,-\pi/2),\ (3,5\pi/2) & \text{in polar coordinates.} \end{array}$$

Note that a negative value of r indicates one moves in the opposite direction from where one would move if r were positive. Because r can be both positive and negative, r is the *directed* distance, and not just the distance, from the origin.

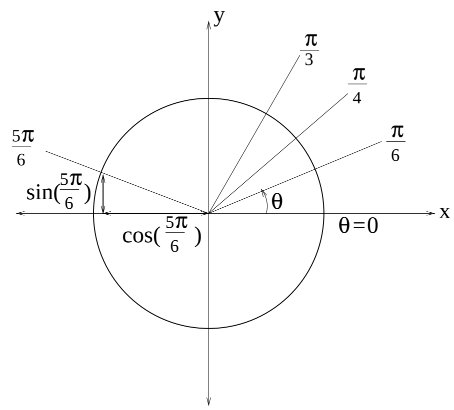
The following formulas are used to convert between coordinates:

$$x = r\cos(\theta) \qquad y = r\sin(\theta) \tag{12}$$

$$r^2 = x^2 + y^2 \qquad \tan(\theta) = \frac{y}{x}. \tag{13}$$

NOTE: The equation $x = r\cos(\theta)$ is NOT equivalent to $\theta = \cos^{-1}(x/r)$ due to the restriction of the range of the inverse function $\cos^{-1}$. See solutions below.

One way to help recall the value of trigonometric functions is to picture the unit circle.



The sine of an angle gives the y-coordinate on the unit circle (since the hypotenuse has length one), and the cosine of an angle gives the x-coordinate. A nice memory device for remembering the sine of major reference angles:

$$\sin(0) = \frac{\sqrt{0}}{2}$$
$$\sin(\frac{\pi}{6}) = \frac{\sqrt{1}}{2}$$
$$\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$$
$$\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$$
$$\sin(\frac{\pi}{2}) = \frac{\sqrt{4}}{2}$$

Solutions to Exercises:

1. Using (12) we have

$$x = r\cos\theta = 2\cos(\frac{5\pi}{3}) = 2(\frac{1}{2}) = 1$$
$$y = r\sin\theta = 2\sin(\frac{5\pi}{3}) = 2(-\frac{\sqrt{3}}{2}) = -\sqrt{3}$$

So our final answer is: $\underline{(x, y) = (1, -\sqrt{3})}$.

2. Using (13) we have

$$r^2 = x^2 + y^2 = \frac{3}{4} + \frac{9}{4} = \frac{12}{4} = 3$$
$$\tan(\theta) = \frac{y}{x} = \frac{3}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

so $r = \sqrt{3}$. Looking at the numbers involved, it's a good guess that $\theta = \frac{\pi}{3}$ or $\frac{\pi}{6}$. Recalling that $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ and doing a quick calculation, we see that the *reference* angle is $\frac{\pi}{6}$. Looking at the rectangular coordinates: $(x, y) = (-\frac{\sqrt{3}}{2}, -\frac{3}{2})$, we see that we need an angle in the *third* quadrant. Thus the angle is $\theta = \frac{7\pi}{6}$. Our final answer is: $\underline{(r, \theta) = (\sqrt{3}, \frac{7\pi}{6})}$. Note that *you do not get this angle by calculating* $\tan^{-1}(\frac{1}{\sqrt{3}})$.

3. Plotting $r = 2$: There are two ways to approach this problem. The first way (which is more laborious, but when all else fails...) is to convert it to cartesian coordinates. To do so, we don't want an r in our equation (which would mean square roots), but r^2. So, squaring both sides, we get $r^2 = 4$ or $x^2 + y^2 = 4$. So this is a circle, centered at the origin, with radius 2. The second way, is to think of the equation directly in terms of polar coordinates: this is the set of all points which are a distance 2 from the origin (for *any* angle θ, the point must be a distance 2 from the origin). This is precisely the definition of a circle centered at the origin with radius 2.

4. Plotting $\theta = \frac{2\pi}{3}$. This means that *for any* r, the angle is $\frac{2\pi}{3}$. Since r may be positive or negative, this is the line through the origin ($r = 0$), at an angle of $\theta = \frac{2\pi}{3}$. (The line should have a negative slope).

5. To convert the equation into polar coordinates, we use (12):

$$\begin{aligned}
&r\cos\theta = r\sin\theta \\
&\cos\theta = \sin\theta \qquad \text{(except at r = 0)} \\
&\tan\theta = 1 \\
&\theta = \frac{\pi}{4}
\end{aligned}$$

so this is a line through the origin at 45 degrees from the positive x-axis (see the discussion in previous solution for plotting $\theta = \frac{2\pi}{3}$). Just to double check, we see that both this equation and the original equation, $x = y$, pass through the origin, so that dividing by r did not affect our results. Another equivalent equation is $\theta = \frac{5\pi}{4}$ - check this.

3 Parametric Equations

In physics, one is often modeling the movement of some object (particle, projectile, car...) through space as a function of time. In this setting, it is not enough to know that a particle moves, for example, in the path of a circle. We need to know *how fast* a particle moves in the circle. To do this, it is more practical to work with equations such as

$$x = f(t) \qquad y = g(t)$$

which are in *parametric form*. Note that this is different from equations that look like $y = h(x)$. In parametric form, we are given more information, since we can determine the position of a particle at any given time, t. You will find this material in your text book in Sections 1.7 and 10.1.

Exercises:

1. Convert the rectangular equation, $y = x^2 + 3$, to a parametric equation.
2. Convert the rectangular equation, $x^2 + y^2 = 9$, to a parametric equation.
3. Convert the parametric equation: $x = 1 - t, \quad y = \sin(t)$,
   to a rectangular equation.
4. Convert the parametric equation: $x = 2\cos(t) \quad y = 3\sin(t)$,
   to a rectangular equation. Sketch it, indicating clearly the direction of the curve for increasing t.

Solution to Exercises:

1. To convert an equation which is in *explicit* form, $y = f(x)$ or $x = g(y)$ (as opposed to an equation in *implicit* form such as the equation of a circle or ellipse), the simplest way is to let one of the variables (x or y) be equal to t. So for example, four parameterizations might be:

$$x = t \qquad y = t^2 + 3 \qquad t \in I\!R \tag{14}$$
$$x = t^2 \qquad y = t^4 + 3 \qquad t \in I\!R \tag{15}$$
$$y = t \qquad x = \pm\sqrt{t-3} \qquad t \geq 3 \tag{16}$$
$$x = \sqrt{t} \qquad y = t + 3 \qquad t \geq 0 \tag{17}$$

   Equations (14) and (15) are valid parameterizations. Notice that a parameterization is *not unique*; that is, the particle could be traveling in a different direction, or at different speeds. We could have let $x = -t$, and then we'd have a parameterization giving a particle going in the opposite direction of the first parameterization above. The third parameterization, (16), is *not valid* because it does not yield two *functions* - for each t, there are two corresponding x-coordinates. The parameterization in (17) gives us only half the curve, so that this parameterization would only be valid for $x \geq 0$.

2. To parameterize a circle or ellipse, using the same techniques as in exercise 1 would result in an expression similar to (17) which is not a valid parameterization because we wouldn't have a single value of x and y for each t. Instead, we take advantage of what we know from polar coordinates, thinking of t as the angle. In this case:

$$x = 3\cos(t) \qquad y = 3\sin(t) \qquad 0 \leq t < 2\pi.$$

This particular parameterization has the particle starting at $(3, 0)$ and traveling the circle counter-clockwise. Other parameterizations are the following:

$$x = 3\sin(t) \qquad y = 3\cos(t)$$
$$x = 3\cos(t) \qquad y = -3\sin(t).$$

Since no restrictions are given on the parameter, t, it is assumed that $-\infty < t < \infty$. To check the parameterizations, substitute the expressions for x and y back into the original equation, $x^2 + y^2 = 9$ and see that it is satisfied identically for all t.

3. For most equations, to convert parametric equations into a rectangular equation, the idea is to eliminate the parameter, t. In this case, $t = 1 - x$, and substituting this into the expression for y we get:

$$y = \sin(1 - x)$$

which is the sine wave offset by 1 in the x-direction. We can check our answer by substituting the parametric equations into the above equation to see that it is satisfied identically for all t.

4. For this set of parametric equations, it is not trivial to eliminate the parameter. But we can make use of the identity $\sin^2(t) + \cos^2(t) = 1$. In this case we have $\cos(t) = x/2$ and $\sin(t) = y/3$, so that we have:

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$
$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

which is the equation of an ellipse.

To sketch it directly from $x = 2\cos(t)$ $y = 3\sin(t)$, we first recognize that this is an ellipse. Then we collect data points, always in the direction of increasing t so that we know the direction: When $t = 0$, $x = 2$, $y = 0$; when $t = \frac{\pi}{2}$, $x = 0$, $y = 3$; when $t = \pi$, $x = -2$, $y = 0$; and continuing, we get an ellipse with vertices $(\pm 2, 0)$, $(0, \pm 3)$, and the direction is counter-clockwise.

4 Differentiation and Max/Min

In this course we will extend the concept of differentiation into multiple dimensions. To do this, it is *absolutely necessary* that you be able to differentiate a function of a single variable and understand how it can be used to find the maximum and minumum of functions. What follows is a very brief *review*. If you find this is not enough of a review, please turn to your text, Chapter 3, Sections 1, 2, 3, 4 and 5; and Section 4.2.

Exercises:

1. Find the derivative of $y = x^3 \sin(x)$.
2. Find the derivative of $y = \frac{\ln(x)}{\cos(x)}$.
3. Find the derivative of $y = \ln(\sin(e^{2x}))$.
4. What is the maximum value of the function $y = \frac{x^2}{1+x}$ on the interval $0 \leq x \leq 5$?

Discussion:
It is expected that you know, without looking at a table, the following differentiation rules:

$$\frac{d}{dx}\left[(kx)^n\right] = kn(kx)^{n-1} \tag{18}$$

$$\frac{d}{dx}\left[e^{kx}\right] = ke^{kx} \tag{19}$$

$$\frac{d}{dx}\left[\ln(kx)\right] = \frac{1}{x} \tag{20}$$

$$\frac{d}{dx}\left[\sin(kx)\right] = k\cos kx \tag{21}$$

$$\frac{d}{dx}\left[\cos(kx)\right] = -k\sin x \tag{22}$$

$$\frac{d}{dx}\left[uv\right] = u'v + uv' \tag{23}$$

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{u'v - uv'}{v^2} \tag{24}$$

$$\frac{d}{dx}\left[u(v(x))\right] = u'(v)v'(x). \tag{25}$$

We put in the constant k into (18) - (22) because a very common mistake to make is something like: $\frac{d}{dx}e^{2x} = \frac{e^{2x}}{2}$ (when the correct answer is $2e^{2x}$). Equation (23) is known as the product rule, Equation (24) is known as the quotient rule, and Equation (25) is known as the chain rule. From these, you can derive the derivative of many other functions, such as the tangent:

$$\begin{aligned}
\tan(x) &= \frac{\sin(x)}{\cos(x)} \\
\frac{d}{dx}\left[\tan(x)\right] &= \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{(\cos(x))^2} \\
&= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}
\end{aligned}$$

$$= \frac{1}{\cos^2(x)}$$
$$= \sec^2(x)$$

where we have used the quotient rule and simplified.

The chain rule is applied when there is a *function of a function*, i.e. $f(g(x))$. The idea is to take the derivative of the outside function first, leaving its argument alone. Then multiply that by the derivative of the next outermost function, leaving it's argument alone. The process is repeated until there is nothing left of which to take the derivative. So for example, to take the derivative of $\sin^2(5x)$, we need to first determine the outside function. If we re-write it as $(\sin(5x))^2$ it is quickly determined that the outside function is "something squared", where "something" in this case is $\sin(5x)$. The derivative of "something squared" is "2 times that something times the derivative of that something". Thus we have

$$\begin{aligned}
\tfrac{d}{dx}\left[\sin^2(5x)\right] &= \frac{d}{dx}\left[(\sin(5x))^2\right] \\
&= 2\sin(5x)\frac{d}{dx}\left[\sin(5x)\right] \\
&= 2\sin(5x)\cos(5x)\frac{d}{dx}\left[5x\right] \\
&= 2\sin(5x)\cos(5x)5 \\
&= 10\sin(5x)\cos(5x)
\end{aligned}$$

So what does the derivative mean? Graphically the derivative gives you the slope of the tangent line for any given value of x (assuming the tangent line exists). This is very useful in helping us determine where a function has its maximum and minimum. If we find where the slope of the tangent line is zero - i.e. where the derivative is zero, then we have found all potential places where the function smoothly turns from increasing to decreasing or vice versa. This is called a *local max/min.* Note that just because the slope is zero does not mean there is necessarily a local max/min at that point; it could be a point of inflection (think $y = x^3$). However, if there is a max/min at a point where the function smoothly changes from increasing to decreasing or vice versa, then it is necessarily true that the slope of the tangent line is zero. There are two types of max/min problems:

1. **Find the maximum or minimum of a function over an open interval**. In this case we are usually dealing with finding the absolute maximum or minimum over an open interval, say from $-\infty$ to ∞. In this case, the absolute maximum/minimum can occur *only where the derivative is zero* (zero tangent line) *or where the derivative is undefined* (the function could go to $+\infty$ for example, or, as in the function $|x|$ at $x = 0$, the derivative may not even exist). These are known as *critical points.* So the idea is to take the derivative, find all places where the derivative is zero or undefined, and check those places *in the original function.*

2. **Find the maximum or minimum of a function over a closed interval**. In this case, the absolute maximum or minimum can occur (i) where there are horizontal

tangents within the closed interval (where the derivative is zero), (ii) where there is a vertical tangent line or where the derivative does not exist (where the derivative is undefined), *or* (iii) at the endpoints. Thus the idea is to find the derivative, determine the points *within the interval* where the derivative is zero or not defined, and then check these points *and the endpoints of the interval* by plugging these values into the original equation.

The only difference between these two problems is that in the second case one must check the endpoints to determine the absolute maximum or minimum.

Solution to Exercises:

1. For this problem, we need the product rule, (23), since two functions are being multiplied. In this case, $u(x) = x^3$ and $v(x) = \sin(x)$. Thus,

$$\begin{aligned} \frac{d}{dx}\left[x^3 \sin(x)\right] &= 3x^2 \sin(x) + x^3 \cos(x) \\ &= 3x^2 \sin(x) + x^3 \cos(x). \end{aligned}$$

2. This is clearly a quotient of functions, so that the quotient rule applies, (24). We have $u(x) = \ln(x)$ and $v(x) = \cos(x)$, which implies:

$$\begin{aligned} \frac{d}{dx}\left[\frac{\ln(x)}{\cos(x)}\right] &= \frac{\frac{1}{x}\cos(x) - \ln(x)[-\sin(x)]}{(\cos(x))^2} \\ &= \frac{\frac{1}{x}\cos(x) + \ln(x)\sin(x)}{\cos^2(x)} \\ &= \frac{\cos(x) + x\ln(x)\sin(x)}{x\cos^2(x)} \end{aligned}$$

3. This is a case of a function of a function of a function of a function, $f(g(h(i(x))))$. We apply the chain rule, always working from the outside function in. In this case the (very) outside function is $f() = \ln$ of "something"; the next most outside function is, $g() = \sin$ of "something", the next most outside function is, $h() = e$ to the "something", and the inside most function is $i(x) = 2x$. Applying the chain rule (25) we have

$$\begin{aligned} \frac{d}{dx}\left[\ln(\sin(e^{2x}))\right] &= \frac{1}{\sin(e^{2x})}\frac{d}{dx}\left[\sin(e^{2x})\right] \\ &= \frac{1}{\sin(e^{2x})}\cos(e^{2x})\frac{d}{dx}\left[e^{2x}\right] \\ &= \frac{1}{\sin(e^{2x})}\cos(e^{2x})e^{2x}\frac{d}{dx}[2x] \\ &= \frac{1}{\sin(e^{2x})}\cos(e^{2x})e^{2x}2 \end{aligned}$$

$$
\begin{aligned}
&= \frac{2e^{2x}\cos(e^{2x})}{\sin(e^{2x})} \\
&= 2e^{2x}\cot(e^{2x})
\end{aligned}
$$

4. We want to find the largest y-value of this function over the interval $0 \leq x \leq 5$. The function can be maximum either where its tangent line has slope zero, where the tangent line is undefined or vertical, or at the endpoints of the interval, $x = 0$ or $x = 5$. So first, let's find the derivative (using the quotient rule, (24)):

$$
\begin{aligned}
\frac{d}{dx}\left[\frac{x^2}{1+x}\right] &= \frac{2x(1+x) - x^2(1)}{(1+x)^2} \\
&= \frac{x(2+x)}{(1+x)^2}.
\end{aligned}
$$

The function has a tangent line of slope zero where the derivative is zero, which occurs when the numerator is zero (and the denominator is not zero), i.e. when $x(2+x) = 0$ or when $x = 0$ or when $x = -2$. Since we do not need to check values of x outside our intervals, we keep only $x = 0$ in our to-check-list.

The function has an undefined derivative when the denominator is zero (and the numerator is not zero). This occurs when $1 + x = 0$, or when $x = -1$. This is outside our interval, so we do not need to check this (critical) point.

Thus the only place where the function may obtain its maximum value(s) are at the critical points within the closed interval, $x = 0$, or at the endpoints, $x = 0$ and $x = 5$. Plugging into the original expression (we want the y-value, if we plug into the derivative we get the slope, not the y-value), we have

$$
\begin{aligned}
f(0) &= \frac{0^2}{1+0} = 0 \\
f(5) &= \frac{5^2}{1+5} = \frac{25}{6}.
\end{aligned}
$$

The maximum value over the interval $0 \leq x \leq 5$ is $y = \frac{25}{6}$.

5 Integration and Area Under the Curve

Like differentiation, in this course we will extend the concept of integration into multiple dimensions, and it is *absolutely necessary* that you be able to integrate a function of a single variable and relate it to finding the area under a curve. What follows is a brief review. If you need supplemental material, please see your text, Sections 5.1, 5.2, 5.3, 5.4, 5.5, 5.6, 5.7, 6.1 and 6.2.

Exercises:

1. Evaluate $\int x\sqrt{x^2+1}\, dx$.

2. Evaluate $\int \frac{\sin(x)}{\cos(x)}\, dx$.

3. Evaluate $\int xe^{3x}\, dx$

4. Given $\int u^n \ln(u)\, du = \frac{u^{n+1}\ln(u)}{n+1} - \frac{u^{n+1}}{(n+1)^2} + C$, evaluate $\int x^2 \ln(2x)\, dx$.

5. Find the area of the region bounded by the curves $y = 8 - x^2$ and $y = 2x$.

The discussion section is rather long, and the solution to these exercises are given at the end of this section.

Discussion:

5.1 Basic Integration Formulas

Not only is it important to be familiar with various integration techniques, but it is also important that we be quick and efficient when evaluating integrals so that we can concentrate on the concepts as oppose to the mechanics of integral problems.

Examples

1. We can evaluate the indefinite integral $\int e^{2x} dx$ by doing a *u-substitution*. However, we can become more efficient at evaluating integrals of this type by obtaining a general formula. Let $f(x) = e^{ax}$, where a is equal to a constant. We would like to obtain a general formula for $\int e^{ax} dx$. We can accomplish this by doing a u-substitution. Let $u = ax$, then $du = adx \Rightarrow \frac{du}{a} = dx$. Substitution yields

$$\int e^{ax} dx = \tfrac{1}{a} \int e^u du = \tfrac{1}{a} e^u + C = \tfrac{1}{a} e^{ax} + C$$

Now we have a general formula that we can use again and again without going to the trouble of doing the u-substitution each time. For example,

(a) $y(x) = e^{2x}$

$$\int e^{2x} dx \;=\; \frac{1}{2} e^{2x} + C$$

(b) $y(x) = e^{\pi x}$

$$\int e^{\pi x} dx = \frac{1}{\pi} e^{\pi x} + C$$

2. Let us follow the procedure in *Example 1* to find the general formula for integrals of the form $\int \cos{(ax)}dx$, where a is equal to a constant. Let $u = ax$, then $du = adx \Rightarrow \frac{du}{a} = dx$. Substitution yields

$$\int \cos axdx = \tfrac{1}{a} \int \cos udu = \tfrac{1}{a} \sin u + C = \tfrac{1}{a} \sin ax + C.$$

Here are some examples of using this general formula.

(a) $y(x) = \cos 4x$

$$\int \cos 4xdx = \frac{1}{4} \sin 4x + C$$

(b) $y(x) = \cos \frac{1}{2\pi} x$

$$\int \cos\left(\frac{1}{2\pi} x\right) dx = 2\pi \sin 2\pi x + C$$

Exercises

Follow the examples above to obtain a general formula for the integral given, then use it to evaluate parts (a) and (b). As above, a is equal to a constant.

1. $\int \sin axdx$
   - (a) $\int \sin 16xdx$
   - (b) $\int \sin \frac{1}{2} xdx$
2. $\int \ln axdx$
   - (a) $\int \ln \pi xdx$
   - (b) $\int \ln \frac{1}{\pi} xdx$
3. $\int \tan axdx$
   - (a) $\int \tan 3xdx$
   - (b) $\int \tan \frac{1}{3} xdx$
4. $\int \sec axdx$

(a) $\int \sec 2.78x dx$

(b) $\int \sec 1618x dx$

5. $\int \arctan{(ax)} dx$

   (a) $\int \arctan \pi x dx$

   (b) $\int \arctan \frac{1}{\pi} x dx$

5.2 u-substitution

In general, u-substitutions are not as straight forward as the ones in the previous section. When doing a u-substitution you want to look for the part of the integral whose derivative is elsewhere in the integral (up to a constant). Formally, if we have an integral of the form

$$\int f(g(x))g^{'}(x)dx,$$

we let $u = g(x)$, then $du = g^{'}(x)dx$, substitution yields

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Essentially, we have transformed the space in which we are evaluating the integral. We evaluate the integral in this new space and then substitute u back in to obtain a solution in the original space. Similar techniques are often employed to solve differential equations.

Examples

1. $\int x^5 e^{x^6} dx$
   First, let $u = x^6$, then $du = 6x^5 \Rightarrow \frac{du}{6} = x^5$. Substitution yields

$$\begin{aligned} \int x^5 e^{x^6} dx &= \frac{1}{6} \int e^u du \\ &= \frac{1}{6} e^u + C \\ &= \frac{1}{6} e^{x^6} + C. \end{aligned}$$

2. $\int (x^2 + 1)^2 (2x) dx$
   First, let $u = x^2 + 1$, then $du = 2x dx$. Substitution yields

$$\begin{aligned}\int (x^2+1)^2(2x)dx &= \int u^2 du \\ &= \frac{1}{3}u^3 + C \\ &= \frac{1}{3}(x^2+1)^3 + C.\end{aligned}$$

Exercises

Use u-substitution to evaluate the following indefinite integrals.

1. $\int \sin^2 3x \cos 3x dx$
2. $\int \frac{1}{\theta^2} \cos \frac{1}{\theta} d\theta$
3. $\int \frac{\sin x}{\cos^2 x} dx$
4. $\int e^x (e^x + 1)^2 dx$
5. $\int \tan^4 x \sec^2 x dx$

5.3 Integration by Parts

Integration by parts is applicable to a plethora of functions which we may need to integrate. Formally, if u and v are functions of x and have continuous derivatives, then

$$\int u dv = uv - \int v du$$

Choosing which part is equal to u may be facilitated by remembering the acronym: LIATE, which stands for: Logarithm, Inverse trig, Algebraic, Trigonometric, Exponential. This means that whichever of these expressions appears first in the acronym, that is the expression you should let u be. So if you want to evaluate $\int (x^2 + 5x - 2)e^{5x}\, dx$, we see we have an algebraic expression, $x^2 + 5x - 2$, times an exponential function, e^{5x}. By this acronym, since A appears before E, we set $u = x^2 + 5x - 2$.

Examples

1. Evaluate $\int xe^x dx$
   First, let $u = x \Rightarrow du = dx$ and let $dv = e^x dx \Rightarrow v = e^x$. Using the integration by parts formula we obtain

$$\begin{aligned} \int xe^x dx &= xe^x - \int e^x dx \\ &= xe^x - e^x + C. \end{aligned}$$

2. Evaluate $\int \arcsin x dx$
   First, let $u = \arcsin x \Rightarrow du = \frac{1}{\sqrt{1-x^2}} dx$ and let $dv = dx \Rightarrow v = x$.

$$\begin{aligned} \int \arcsin x dx &= x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx \\ &= x \arcsin x + \frac{1}{2} \int w^{-\frac{1}{2}} dw \\ &= x \arcsin x + w^{\frac{1}{2}} + C \\ &= x \arcsin x + (1-x^2)^{\frac{1}{2}} + C \\ &= x \arcsin x + \sqrt{1-x^2} + C. \end{aligned}$$

   Where the second equality comes from doing a $u-$*substitution* (w in this case) where $w = 1 - x^2$

3. Sometimes it is necessary to do integration by parts more than once. For example, $\int x^2 e^x dx$.
   First, let $u = x^2 \Rightarrow du = 2x dx$ and let $dv = e^x dx \Rightarrow v = e^x$. Substitution yields

$$\begin{aligned} \int x^2 e^x dx &= x^2 e^x - 2 \int xe^x dx \\ &= x^2 e^x - 2(xe^x - e^x) + C \\ &= x^2 e^x - 2xe^x - 2e^x + C. \end{aligned}$$

   where the second equality comes from our previous calculation in *Example 1*.

4. Here is another example where integration by parts will be used repeatedly to evaluate an integral. Evaluate $y(x) = e^x \cos x$.
   First, let $u = e^x \Rightarrow du = e^x dx$ and let $dv = \cos x dx \Rightarrow v = \sin x$. Substitution yields

$$\int e^x \sin x dx = e^x \sin x - \int e^x \sin x dx.$$

   Using integration by parts again, let $u = e^x \Rightarrow du = e^x dx$ and let $dv = \sin x dx \Rightarrow v = -\cos x$. Substitution yields

$$
\begin{aligned}
\int e^x \cos x dx &= e^x \sin x - \left[-e^x \cos x + \int e^x \cos x dx\right] \\
&= e^x \sin x + e^x \cos x - \int e^x \cos x dx \\
2\int e^x \cos x dx &= e^x \sin x + e^x \cos x \\
\int e^x \cos x dx &= \frac{1}{2}(e^x \sin x + e^x \cos x) + C.
\end{aligned}
$$

Exercises
Use integration by parts to evaluate the following indefinite integrals.

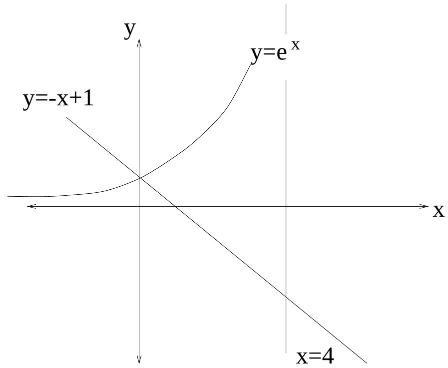
1. $\int t \ln(t+1) dt$
2. $\int \frac{(\ln x)^2}{x} dx$
3. $\int \arccos x dx$
4. $\int e^{2x} \sin x dx$
5. $\int e^x \sin x dx$

5.4 Finding the area between curves

. In Calculus 3, we'll be extending the idea of calculating area within a plane, to calculating area of 3-dimensional figures and calculating volume. Here we review the ideas behind calculating area. Consider the following problem:
sample problem: Find the area of the region bounded by:

$$x = 4 \quad y = e^x, \quad y = -x + 1.$$

Note that without some sketching it's difficult to see what in the world this region looks like and where the bounds should be set when integrating. You should know what $y = e^x$ looks like. $y = -x + 1$ is the equation of a line with a negative slope, and when $x = 0$, $y = 1$, and when $y = 0$, $x = 1$. Thus a quick sketch reveals:



So the intersection of the two curves, $y = e^x$ and $y = -x + 1$, occurs at $(0, 1)$. Note that without the sketch it would not be so easy to find this point of intersection - setting the two y-coordinates equal yields the equation $e^x = -x + 1$ which is not easily solvable (except perhaps by inspection - when $x = 0$ this equation is satisfied identically). The area bounded by the 3 curves is then given by:

$$\begin{aligned} \int_0^4 e^x - (-x+1)\, dx &= \int_0^4 (e^x + x - 1)\, dx \\ &= e^x + \frac{1}{2}x^2 - x\Big|_0^4 \\ &= e^4 + 8 - 4 - (1 + 0 - 0) \\ &= e^4 + 3 \end{aligned}$$

If you have a calculator available, one sees that this is approximately equal to 57.6.

Solutions to Exercises:

1. This requires some substitution. Our best hope is to let $u = x^2 + 1$, since this is what is under the square root (and it is very unlikely that this is part of a du). With this choice of u, we have $du = 2x$. We have the x, but not the 2, but since 2 is a constant, it is very easily taken care of. Multiplying by $\left(\frac{1}{2}\right)(2)$ which doesn't change our problem since we are multiplying by 1, we get

$$\begin{aligned} \int x\sqrt{x^2+1}\, dx &= \frac{1}{2}\int \sqrt{x^2+1}\, 2xdx \\ &= \frac{1}{2}\int \sqrt{u}\, du \\ &= \frac{1}{2}\int u^{\frac{1}{2}}\, du \\ &= \frac{1}{2}\frac{2}{3}u^{\frac{3}{2}} + C \\ &= \frac{1}{3}(x^2+1)^{\frac{3}{2}} + C \end{aligned}$$

We can check our result by differentiating.

2. This again requires some substitution. We can try $u = \sin(x)$ or $u = \cos(x)$. Trying both ways, it becomes clear that the only workable solution is $u = \cos(x)$, so that we have a form of $\frac{1}{u}\ du$. In this case, ($u = \cos(x)$) we get $du = -\sin(x)\ dx$ so that

$$\begin{aligned}\int \frac{\sin(x)}{\cos(x)}\ dx &= -\int \frac{1}{\cos(x)}\ (-\sin(x))\ dx \\ &= -\int \frac{1}{u}\ du \\ &= -\ln(u) + C \\ &= -\ln(\cos(x)) + C\end{aligned}$$

Again, we can check our answer by differentiating.

3. This requires the use of integration by parts. Thinking LIATE, let $u = x$ and $dv = e^{3x}\ dx$. Then

$$\begin{aligned}\int xe^{3x}\ dx &= x\frac{1}{3}e^{3x} - \int \frac{1}{3}e^{3x}\ dx \\ &= x\frac{1}{3}e^{3x} - \frac{1}{9}e^{3x}\end{aligned}$$

4. Here $n = 2$, $u = 2x$, so $du = 2\ dx$. We don't have quite the right du in its current form, nor do we have $u^2 = (2x)^2$, so we must multiply and divide by 8:

$$\begin{aligned}\int x^2 \ln(2x)\ dx &= \frac{1}{8}\int (2x)^2 \ln(2x) 2\ dx \\ &= \frac{1}{8}\int u^2 \ln(u); du \\ &= \frac{1}{8}\left[\frac{u^3 \ln(u)}{3} - \frac{u^3}{3^2}\right] + C \\ &= \frac{1}{8}\left[\frac{(2x)^3 \ln(2x)}{3} - \frac{(2x)^3}{9}\right] + C \\ &= \frac{1}{8}\left[\frac{8}{3}x^3 \ln(2x) - \frac{8}{9}x^3\right] + C \\ &= \frac{1}{3}x^3 \ln(2x) - \frac{1}{9}x^3 + C\end{aligned}$$

5. It may help to sketch these to curves to get an idea of what we're looking at: $y = 8 - x^2$ is a parabola with a vertex at $(0, 8)$ and which opens down. $y = 2x$ is a line with positive slope going through the origin. The intersection occurs:

$$\begin{aligned}&8 - x^2 = 2x \\ &x^2 + 2x - 8 = 0 \\ &(x - 2)(x + 4) = 0 \\ &x = 2 \text{ or } x = -4.\end{aligned}$$

The parabola lies on top (thinking of sketch), and we need to evaluate:

$$\begin{aligned}
\int_{-4}^{2} (8 - x^2 - 2x)\, dx &= 8x - \frac{x^3}{3} - x^2 \Big]_{-4}^{2} \\
&= 8(2) - \frac{8}{3} - 4 - \left[8(-4) - \frac{(-4)^3}{3} - 16 \right] \\
&= 4\left[4 - \frac{2}{3} - 1 + 8 - \frac{16}{3} + 4 \right] \\
&= 4\left[15 - \frac{18}{3} \right] \\
&= 4\,[15 - 6] \\
&= 4[9] \\
&= 36
\end{aligned}$$