Standard Note Sheet – MATH 2421

1. 2D & 3D Vectors

   The unit direction vector associated with nonzero $\mathbf{u}$ is $\dfrac{\mathbf{u}}{\|\mathbf{u}\|}$.

   $\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\| * \|\mathbf{v}\| * \cos(\alpha)$. The angle of separation is α.

   $\mathbf{u}\cdot\mathbf{u} = \|\mathbf{u}\|^2$. $\quad \mathbf{u}\cdot\mathbf{v} = 0 \Rightarrow \mathbf{u}$ and $\mathbf{v}$ are orthogonal.

   $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\| * \|\mathbf{v}\| * \sin(\alpha)$. The angle of separation is α.

   If a 3D line passes through $P(x_0, y_0, z_0)$ with a direction vector $\mathbf{v} = \langle a, b, c\rangle$, then the parametric form for the line is:

$$\begin{aligned} x &= at + x_0 \\ y &= bt + y_0 \\ z &= ct + z_0 \end{aligned}$$

2. Surfaces

   If a plane in space has normal vector $\mathbf{n} = \langle a, b, c\rangle$ and it passes through the point $P(x_0, y_0, z_0)$, then its standard form (in Rectangular Coordinates) is:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

   $x^2 + y^2 = R^2$ is a right circular cylinder perpendicular to the xy-plane.

   $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$. Ellipsoid. $\qquad z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$. Elliptic paraboloid.

   $z = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$. Hyperbolic paraboloid (saddle). $\qquad z = \sqrt{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}}$. Elliptic (upper) cone.

   $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$. Hyperboloid of ONE sheet. (One negative sign.)

3. Vector-valued Functions and Arc Length

   If the position function is $\mathbf{r}(t) = \langle x(t),\ y(t),\ z(t)\rangle$, then

$$ds = \|\mathbf{v}(t)\|\, dt = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt.$$

   Unit Tangent Vector: $\mathbf{T}(t) = \dfrac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|}$, $\mathbf{v}(t) \neq \mathbf{0}$.

   Principal Unit Normal Vector: $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$, $\mathbf{T}'(t) \neq \mathbf{0}$. [Always points *into* the curve.]

   Curvature: $\kappa = \dfrac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$. The radius of curvature is $\dfrac{1}{\kappa}$.

4. Multivariable Functions and Partial Derivatives

   Total differential of $z = f(x, y)$: $\quad dz = f_x dx + f_y dy$.

   Leibniz form of the Multivariable Chain Rule:
   [If $z = f(x, y)$, $x = x(s, t)$, and $y = y(s, t)$, then...]

$$\frac{\partial z}{\partial t} = \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial x}{\partial t}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial y}{\partial t}\right)$$

   This also works for related rates problems.

   Implicit differentation: If $F(x, y, z) = 0$, then $\dfrac{\partial z}{\partial x} = -\left(\dfrac{F_x}{F_z}\right)$.

   $\nabla f(x, y, z) = \langle f_x,\ f_y,\ f_z \rangle$. The gradient vector always points in the direction of the greatest directional derivative (best rate of increase in f).

   If we have a surface $F(x, y, z) = 0$, which is a level surface of the parent function F, then ∇F is always normal to that surface, and thus, it must also be normal to the plane tangent to that surface.

   If $\nabla f = \mathbf{0}$ or any of the components of ∇f is undefined, then we have a critical point.
   Second Partials Test on those critical points: $d = f_{xx} f_{yy} - (f_{xy})^2$.
   If $d > 0$ and $f_{xx} > 0$, we have a local minimum.
   If $d > 0$ and $f_{xx} < 0$, we have a local maximum.
   If $d < 0$, we have a saddle point. If $d = 0$, the test is inconclusive.

5. Conversions

   (a) Rectangular to Polar/Cylindrical:
   $x = r\cos(\theta) \quad y = r\sin(\theta) \quad \dfrac{y}{x} = \tan(\theta) \quad x^2 + y^2 = r^2$.

   (b) Polar/Cylindrical to Rectangular:
   $r = \sqrt{x^2 + y^2} \quad \tan(\theta) = \dfrac{y}{x}$.

   (c) Cylindrical to Spherical:
   $z = \rho\cos(\phi) \quad r = \rho\sin(\phi)$

   (d) Rectangular to Spherical:
   $x = \rho\sin(\phi)\cos(\theta) \quad y = \rho\sin(\phi)\sin(\theta)$

   (e) Spherical to Rectangular:
   $\rho = \sqrt{x^2 + y^2 + z^2} \quad \rho^2 = x^2 + y^2 + z^2 \quad \cos(\phi) = \dfrac{z}{\sqrt{x^2 + y^2 + z^2}}$.

   $r = a\ constant$ is a right circular cylinder.
   $\rho = a\ constant$ is a sphere when $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$.
   $\phi = a\ constant\ angle$ is a cone when $\rho \ge 0$ and $0 \le \theta \le 2\pi$.

6. "Del" operator $= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$ is a vector.

   $\nabla f(x,y,z) = \langle f_x, \ f_y, \ f_z \rangle$ is a gradient vector field.

   $\nabla \cdot \langle M, \ N, \ P \rangle = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle M, \ N, \ P \rangle = M_x + N_y + P_z =$ Divergence.

   $\nabla \times \langle M, \ N, \ P \rangle = \langle P_y - N_z, \ M_z - P_x, \ N_x - M_y \rangle =$ Curl.

7. Area and Volume Differentials

   $dA = dy \ dx = r \ dr \ d\theta$

   $dV = dz \ dy \ dx = dz \ r \ dr \ d\theta = \rho^2 \sin(\phi) \ d\rho \ d\phi \ d\theta$

8. Surface Area

   If $z = f(x,y)$, then

   $$dS = \sqrt{1 + (f_x)^2 + (f_y)^2} \ dA.$$

9. Line Integration

   $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M dx + N dy + P dz = \int_{t_1}^{t_2} \mathbf{F}(x(t), \ y(t), \ z(t)) \cdot \mathbf{v}(t) \, dt$$

   $$mass = \int_C \sigma(x,y) \, ds.$$

10. Conservative Field

    A 2D vector field $\mathbf{F}(x,y) = \langle M, \ N \rangle$ is conservative if and only if $N_x - M_y = 0$.

    A 3D vector field $\mathbf{F}(x,y,z) = \langle M, \ N, \ P \rangle$ is conservative if and only if $\nabla \times \mathbf{F} = \langle 0, 0, 0 \rangle$.

    If a vector field $\mathbf{F}$ is conservative, then the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ only depends on the starting point and the ending point of the curve C (path independent).

    Furthermore, there must be a potential (scalar) function f such that $\nabla f = \mathbf{F}$ and the value of the line integral must be

    $$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\textit{ending point}) - f(\textit{starting point}).$$

11. Green's Theorem

    If C is a simple closed curve and R is the interior of C, then

    $$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_C M dx + N dy = \iint_R (N_x - M_y) \ dA,$$

    where the closed line integral is traced once around C counterclockwise.

12. Divergence Theorem

    If S is a simple closed surface and Q is its interior, then the net outward flux through S is

    $$\oint_S (\mathbf{F} \cdot \mathbf{n}) \, dS = \iiint_Q (\nabla \cdot \mathbf{F}) \, dV.$$

    It is assumed that $\mathbf{n}$ is the outward unit normal vector on S.