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Sov. J. Numer. Anal. Math. Modelling, Vol. 5, Nos. 4/5, pp. 265-275 (1990)
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The Temple-Lehmann-type methods in iterative algorithms

A. V. KNYAZEV, V. I. LEBEDEV and A. L. SKOROKHODOV

Abstract – Exact *a priori* error estimates are obtained for two-sided approximations to the lower eigenvalues of a symmetric matrix, computed by the Temple–Lehmann-type methods. An *a priori* optimization of parameters is carried out in iterative methods of such type.

We intend to develop iterative versions of the Temple–Lehmann method for obtaining two-sided approximations to the lower eigenvalues of a symmetric matrix A. Let us introduce a Temple functional [2,8]

$$\lambda^{\alpha}(\cdot) \equiv (A\cdot,(A-\alpha)\cdot)/((A-\alpha)\cdot,\cdot)$$

where α is a real parameter. It was shown in [2] that under an appropriate choice of the parameter α and the vector u the quantity $\lambda^{\alpha}(u)$ can estimate any eigenvalue both from above and from below. In [6,7], N. J. Lehmann suggested that the stationary values of the functional λ^{α} be computed on a 'trial' subspace U by analogy to the Rayleigh–Ritz method and proved that these values provide two-sided estimates simultaneously for a group of eigenvalues with appropriately chosen α and U. In [8], N. J. Lehmann pointed to the possibility of choosing the Krylov subspace for U and considered the iterative realization of the method.

In this paper, we suggest instead of $\lambda^{\alpha}(\cdot)$ a functional of more general form, obtain exact error estimates (cf. [3]) and to compute the stationary value of this new functional, we construct iterative algorithms of the subspace iteration type [3,4] with the optimal *a priori* choice of iterative parameters.

1. TWO-SIDED ESTIMATES FOR A LOWER EIGENVALUE

In an Euclidean space H let us consider the following eigenvalue problem:

$$Au = \lambda u, \quad A = A^{*}. \tag{1.1}$$

Number the eigenvalues of the operator A in the increasing order: $\lambda_1 < \lambda_2 \leqslant \ldots \leqslant \lambda_{\max}$ (without loss of generality, λ_1 is assumed to be simple) and denote the corresponding orthonormal eigenvectors by $u_1, u_2, \ldots, u_{\max}$. Let the numbers $\underline{\lambda} \in (\lambda_1, \lambda_2]$ and $\bar{\lambda} \geqslant \lambda_{\max}$ be known and on the set $\lambda_1 \cup [\underline{\lambda}, \bar{\lambda}]$ a real-valued function $Q(\lambda)$ be given which is continuous on the segment $[\underline{\lambda}, \bar{\lambda}]$ and such that $Q(\lambda_1) > 0$. Let us introduce the quantity σ:

$$\sigma \equiv \sigma(Q) = \max_{\lambda \in [\underline{\lambda}, \bar{\lambda}]} |Q(\lambda)|/Q(\lambda_1)$$

and define the functional

$$\lambda^{Q}(u) \equiv \frac{(AQ(A)u,u)}{\ldots}, \quad u \in H \backslash 0.$$

We have $\lambda^Q(u_1) = \lambda_1$. Assume the value of the functional $\lambda^Q(u)$ to be an approximation to λ_1. Let us formulate the error estimates.

Theorem 1.1. (a) If $Q(\lambda) \geqslant 0$ for $\lambda \in [\underline{\lambda}, \bar{\lambda}]$, then

$$0 \leqslant \frac{\lambda^Q(u) - \lambda_1}{\lambda_{\max} - \lambda^Q(u)} \leqslant \sigma \operatorname{tg}^2 \angle(u; u_1). \tag{1.2}$$

(b) If $Q(\lambda) \leqslant 0$ for $\lambda \in [\underline{\lambda}, \bar{\lambda}]$ and

$$\sigma \operatorname{tg}^2 \angle(u; u_1) < 1 \tag{1.3}$$

then

$$0 \leqslant \frac{\lambda_1 - \lambda^Q(u)}{\lambda_{\max} - \lambda^Q(u)} \leqslant \sigma \operatorname{tg}^2 \angle(u; u_1). \tag{1.4}$$

Remark 1.1. [on the attainability of estimates (1.2) and (1.4)]. Estimate (1.2) [or (1.4)] is attainable in the sense that for any positive number c there exist functions $Q(\lambda) \geqslant 0$ [or $Q(\lambda) \leqslant 0$] for $\lambda \in [\underline{\lambda}, \bar{\lambda}]$ and a vector u such that $\sigma \operatorname{tg}^2 \angle(u; u_1) = c$ and the right-hand inequality in (1.2) [or (1.4)] becomes an equality.

Remark 1.2. [on the possibility to make constraint (1.3) weaker]. If $Q(\lambda) \leqslant 0$ for $\lambda \in [\underline{\lambda}, \bar{\lambda}]$, the inequality $\lambda^Q(u) \leqslant \lambda_1$ will also hold with the constraint $(Q(A)u, u) > 0$ weaker than (1.3).

2. AN A PRIORI OPTIMIZATION

Let Π_N^+ and Π_N^- be sets of all polynomials P_N^+ and P_N^- of degree not exceeding N, positive at the point λ_1 and preserving the sign on the segment $[\underline{\lambda}, \bar{\lambda}]$: $P_N^+(\lambda) \geqslant 0$ for $\lambda \in [\underline{\lambda}, \bar{\lambda}]$. Let us solve the problems of minimization of the quantities $\sigma(Q_N^+)$ and $\sigma(Q_N^-)$ on the sets Π_N^+ and Π_N^-, respectively.

Theorem 2.1. (a) The following equalities are valid:

$$\min \sigma(Q_N^+) = \sigma(R_N + 1) = \frac{2}{T_N(\theta) + 1}.$$

(b) The following equalities are valid:

$$\max \sigma(Q_N^-) = \sigma(R_N - 1) = \frac{2}{T_N(\theta) - 1}$$

where

$$R_N \equiv R_N(\lambda) = T_N\left(\frac{\bar{\lambda} + \underline{\lambda} - 2\lambda}{\bar{\lambda} - \underline{\lambda}}\right)$$

$T_N(x)$ is the Nth-degree Chebyshev polynomial of the first kind; and $\theta \equiv (\bar{\lambda} + \underline{\lambda} - 2\lambda_1)/(\bar{\lambda} - \underline{\lambda})$, $\theta > 1$ (see Figs. 1 and 2).

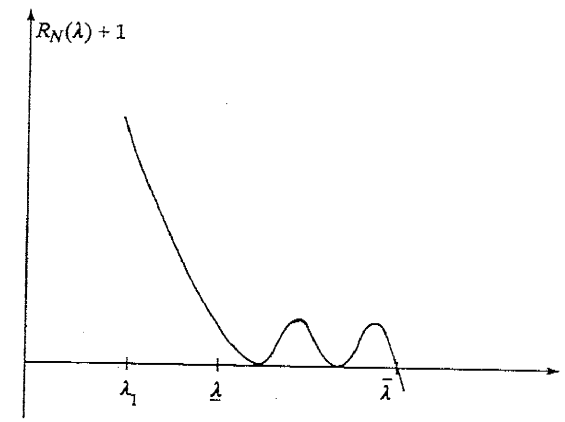

Figure 1. A form of optimal polynomial $R_N(\lambda) + 1$.

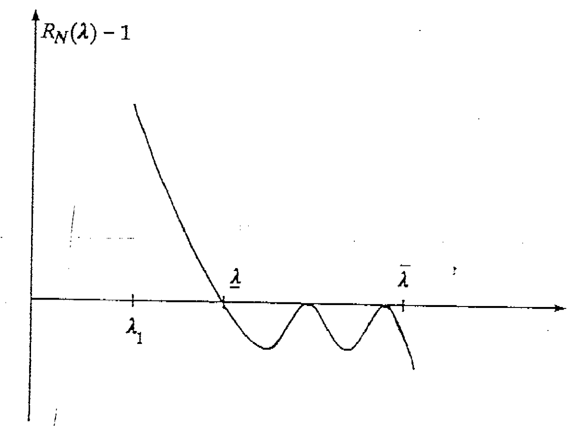

Figure 2. A form of optimal polynomial $R_N(\lambda) - 1$.

Using Theorem 2.1 we can suggest optimal [in the sense of minimization of $\sigma(Q)$] iterative methods for finding approximations to λ_1. Assume that $N = 2n + 1$, $n = 0, 1, \ldots$, and $x \in [-1, 1]$.

Exploiting the relation

$$T_N(x) + 1 = (1 + x) W_n^2(-x)$$

where

$$W_n(x) = \frac{\sin\left(\frac{2n+1}{2} \arccos x\right)}{\sin\left(\frac{1}{2} \arccos x\right)}$$

is a Chebyshev polynomial of the third kind, write down the algorithm for finding quantities of $\lambda^+ \equiv \lambda^{T_{N+1}}(u^0)$-optimal approximations to λ_1 from above:

$$\begin{aligned} v^{k-1} &= Au^{k-1} - \alpha_k u^{k-1} \\ u^k &= v^{k-1}/\|v^{k-1}\|, \quad k = 1,\ldots,n; \\ u^n &= Au^n \\ \lambda^+ &= (v^n, \bar{\lambda}u^n - v^n)(\bar{\lambda}u^n - v^n, u^n). \end{aligned} \tag{2.1}$$

Here, u^0, $\|u^0\| = 1$, is a given vector, $\operatorname{tg}\angle(u^0;u_1) < \infty$ and

$$\alpha_k = \tfrac{1}{2}[\underline{\lambda} + \bar{\lambda} + (\bar{\lambda} - \underline{\lambda})\omega_k]$$

ω_k is a 'W-sequence' of stably ordered roots of the polynomial $W_n(x)$ [5].

Knowing the vectors u^n and v^n from (2.1) and making use of the relation

$$\lambda^- \equiv \lambda^{T_{N-1}}(u^0) = \frac{(A(T_N(A)+I)u^0,u^0) - 2(Au^0,u^0)}{((T_N(A)+I)u^0,u^0) - 2(u^0,u^0)}$$

we can readily find the optimal approximation to λ_1 from below:

$$\lambda^- = \frac{(v^n,\bar{\lambda}u^n - v^n) - 2(Au^0,u^0)/\prod_{k=1}^{n}\|v^{k-1}\|}{(u^n,\bar{\lambda}u^n - v^n) - 2(u^0,u^0)/\prod_{k=1}^{n}\|v^{k-1}\|}.$$

Making use of the formula

$$W_{3n+1}(x) = W_n(x)[2T_{2n+1}(x) + 1]$$

we can also suggest infinitely continued stable iterative methods optimal in this case for $N = 3n+1$, $n = 1,2,\ldots$.

3. TWO-SIDED ESTIMATES FOR SEVERAL LOWER EIGENVALUES

Let us consider the problem of computation of p lower eigenvalues of problem (1.1). Assume (mainly to simplify writing down) the eigenvalues $\lambda_1,\ldots,\lambda_p$ to be simple and the numbers $\underline{\lambda} \in (\lambda_p,\lambda_{p+1}]$, $\tilde{\lambda} \geq \lambda_{max}$ to be known. Let on the set $\{\lambda_1,\ldots,\lambda_p\} \cup [\underline{\lambda},\tilde{\lambda}]$ the real-valued function $Q(\lambda)$ be given which is continuous on the segment $[\underline{\lambda},\tilde{\lambda}]$ and positive at the points $\lambda_1,\ldots,\lambda_p$. Let us introduce the following quantities for $1 \leq j \leq p$:

$$\delta_j \equiv \delta_j(Q) = \max_{\lambda \in [\underline{\lambda},\tilde{\lambda}]} |Q(\lambda)| / \min_{i=1,\ldots,j} Q(\lambda_i)$$

and

$$\sigma_j \equiv \sigma_j(Q) = \max_{\lambda \in [\underline{\lambda},\tilde{\lambda}]} |Q(\lambda)| / \min_{i=1,\ldots,p} Q(\lambda_i)$$

$$\bar{\sigma} = \max_{j=1,\ldots,p} \sigma_j.$$

Denote $U_p \equiv \operatorname{span}\{u_1,\ldots,u_p\}$, U is a subspace in H of dimension p: $\dim U = \dim U_p = p$; $\Phi \equiv \angle(U;U_p)$ is the angle between the subspaces U and U_p. Define on the subspace U the functional $\lambda^Q(u)$:

$$\lambda^Q(u) = (AQ(A)u,u)/(Q(A)u,u), \quad u \in U\backslash 0$$

and denote by λ_j^Q, $j = 1,\ldots,p$, its stationary values. Assume λ_j^Q to be approximations to λ_j and give error estimates for different $Q(\lambda)$ [block counterparts of estimates (1.2) and (1.4)].

Theorem 3.1. Let $Q(\lambda) \geq 0$ for $\lambda \in [\underline{\lambda},\tilde{\lambda}]$. Then each $j = 1,\ldots,p$ satisfies the inequalities

$$0 \leq \frac{\lambda_j^Q - \lambda_j}{\lambda_{max} - \lambda_j^Q} \leq \delta_j \operatorname{tg}^2\Phi. \tag{3.1}$$

Theorem 3.2. Let $Q(\lambda) \leq 0$ for $\lambda \in [\underline{\lambda},\tilde{\lambda}]$ and

$$\bar{\sigma}\operatorname{tg}^2\Phi < 1. \tag{3.2}$$

Then each $j = 1,\ldots,p$ satisfies the inequalities

$$0 \leq \frac{\lambda_j - \lambda_j^Q}{\lambda_{max} - \lambda_j^Q} \leq \sigma_j \operatorname{tg}^2\Phi. \tag{3.3}$$

Now let Π_N^+ and Π_N^- be sets of all polynomials P_N^+ and P_N^- of degree not exceeding N, positive at the points $\lambda_1,\ldots,\lambda_p$ and preserving the sign on the segment $[\underline{\lambda},\tilde{\lambda}]$: $P_N^+(\lambda) \geq 0$, $P_N^-(\lambda) \leq 0$ for $\lambda \in [\underline{\lambda},\tilde{\lambda}]$. Similarly to Theorem 2.1 we can formulate a statement on the form of the polynomials $Q_N^+(\lambda) \in \Pi_N^+$, $Q_N^-(\lambda) \in \Pi_N^-$ minimizing the quantities δ_j and σ_j, $j = 1,\ldots,p$, respectively.

Theorem 3.3. (a) The quantities δ_j simultaneously take minimal values on the polynomial $Q_N^+(\lambda) = R_N(\lambda) + 1$ and $\delta_j = 2/(T_N(\theta_j) + 1)$. (b) The quantities σ_j simultaneously take minimal values equal to $\bar{\sigma}$ on the polynomial $Q_N^-(\lambda) = R_N(\lambda) - 1$ and $\bar{\sigma} = 2/(T_N(\theta_p) - 1)$.

Here,

$$R_N(\lambda) \equiv T_N\left(\frac{\tilde{\lambda} + \underline{\lambda} - 2\lambda}{\tilde{\lambda} - \underline{\lambda}}\right), \quad \theta_j = \frac{\tilde{\lambda} + \underline{\lambda} - 2\lambda_j}{\tilde{\lambda} - \underline{\lambda}}.$$

We can also suggest optimal (in the sense of simultaneous minimization of the quantities δ_j and σ_j) subspace iteration methods for finding two-sided approximations to λ_j, $j = 1,\ldots,p$.

Let U^0 be a given subspace, $\dim U^0 = p$, and its basis be known: $U^0 = \operatorname{span}\{u_1^0,\ldots,u_p^0\}$.

To find approximations to λ_j from above, perform the following computations:

where $\alpha_k = \frac{1}{2}[\tilde{\lambda} + \underline{\lambda} + (\tilde{\lambda} - \underline{\lambda})\omega_k]$. Expression (3.4) means that independent iterations are simultaneously carried out with vectors $u_1^{k-1},\ldots,u_p^{k-1}$, which result in our arriving at the subspace $U^n = \operatorname{span}\{u_1^n,\ldots,u_p^n\}$, $\dim U^n = p$, where $u_j^n = \prod_{k=1}^n (A - \alpha_k I)u_j^0$ and $v_j^n = Au_j^n$, $j = 1,\ldots,p$. Then make up symmetric $p \times p$ matrices $\widehat{AQ(A)} = \{(v_i^n, \tilde{\lambda}u_j^n - v_j^n)\}$ and $\widehat{Q(A)} = \{(u_i^n, \tilde{\lambda}u_j^n - v_j^n)\}$, where $\widehat{Q(A)}$ is positive defnite, and solve the generalized eigenvalue problem

$$\widehat{AQ(A)}y = \lambda^Q \widehat{Q(A)}y. \tag{3.5}$$

The eigenvalues of problem (3.5) coincide with the corresponding stationary values λ_j^Q of the functional $\lambda^Q(u)$, $u \in U^0\backslash 0$, and according to Theorems 3.1 and 3.3 (a) will approximate corresponding λ_j from above in an optimal way.

To find approximations to λ_j from below, we make use of the relation

$$T_N(x) - 1 = (x-1)w_n^2(x), \quad N = 2n+1, \quad n = 0,1,\ldots$$

and compute U^k by the formula

$$U^k = AU^{k-1} - \beta_k U^{k-1}, \quad k = 1,\ldots,n$$

where $\beta_k = \frac{1}{2}[\tilde{\lambda} + \underline{\lambda} - (\tilde{\lambda} - \underline{\lambda})\omega_k]$. Make up as before with the substitution $\tilde{\lambda} \to \underline{\lambda}$ the matrices $\widehat{AQ(A)}$ and $\widehat{Q(A)}$; in this case, if constraint (3.2) is satisfied, the matrix $\widehat{Q(A)}$ is positive definite. Compute the eigenvalues λ_j^Q of problem (3.5) which will be optimal approximations to corresponding λ_j from below.

4. PROOFS

Proof of Theorem 1.1. Define the vectors $w = (u,u_1)u_1$ and $v = u - w$. In the non-trivial case, $v \neq 0$, $Q(A)v \neq 0$, and we have

$$\operatorname{tg}^2 \angle(u;u_1) \equiv \operatorname{tg}^2\varphi = (v,v)/(w,w)$$

$$\lambda^Q(u) = \frac{\lambda_1 Q(\lambda_1)(w,w) + (AQ(A)v,v)}{Q(\lambda_1)(w,w) + (Q(A)v,v)}.$$

(a) Assume that $Q(\lambda) \geq 0$ for $\lambda \in [\underline{\lambda},\bar{\lambda}]$. Then,

$$\lambda^Q(u) - \lambda_1 = \frac{((A - \lambda_1 I)Q(A)v,v)}{(Q(A)u,u)} \geq 0$$

and since $(Q(A)u,u) > 0$ and $(Q(A)v,v) > 0$, then $\lambda^Q(u) < \lambda^Q(v) \leq \lambda_{\max}$, and the following inequalities are valid:

$$\frac{\lambda^Q(u) - \lambda_1}{\lambda_{\max} - \lambda^Q(u)} \leq \frac{\lambda^Q(u) - \lambda_1}{\lambda^Q(v) - \lambda^Q(u)} = \frac{(Q(A)v,v)}{Q(\lambda_1)(w,w)} \leq \sigma \operatorname{tg}^2\varphi.$$

This completes the proof of inequalities (1.2).

(b) Assume that $Q(\lambda) \leq 0$ for $\lambda \in [\underline{\lambda},\bar{\lambda}]$ and constraint (1.3) is satisfied. Then,

$$(Q(A)u,u) = Q(\lambda_1)(w,w) - (|Q(A)|v,v) \geq Q(\lambda_1)(w,w)(1 - \sigma\operatorname{tg}^2\varphi) > 0 \tag{4.1}$$

and

$$\lambda_1 - \lambda^Q(u) = \frac{((A - \lambda_1 I)|Q(A)|v,v)}{(Q(A)u,u)} \geq 0. \tag{4.2}$$

Since $\lambda_{\max} \geq \lambda^Q(v) \geq \lambda^Q(u)$, then

$$\frac{\lambda_1 - \lambda^Q(u)}{\lambda_{\max} - \lambda^Q(u)} \leq \frac{\lambda_1 - \lambda^Q(u)}{\lambda^Q(v) - \lambda^Q(u)} = \frac{(|Q(A)|v,v)}{Q(\lambda_1)(w,w)} \leq \sigma\operatorname{tg}^2\varphi.$$

This completes the proof of inequalities (1.4).

Proof of Remark 1.1. For the right-hand side of inequality (1.2) [or (1.4)] to become an equality, it is sufficient to choose the function $Q^+(\lambda)$ [or $Q^-(\lambda)$]:

$$Q^{\pm}(\lambda) = \begin{cases} 1, & \lambda = \lambda_1 \\ \pm Q, \quad Q > 0, & \lambda \in [\underline{\lambda},\bar{\lambda}] \end{cases}$$

and the vector u of the form $u = u_1 + \sqrt{c/Q}\,u_{\max}$.

The proof of Remark 1.2 is implied by inequalities (4.1), (4.2).

Proof of Theorem 2.1. Let us prove the statement from (a). Assume that there exists a polynomial $Z_N^+ \in \mathbb{P}_N^+$ such that $\sigma(Z_N^+) < \sigma(R_N + 1)$. Then for the polynomial $Z_N^+(\lambda) - 1$ we have $\sigma(Z_N^+ - 1) < \sigma(R_N) = T_N^{-1}(\theta)$ and it is impossible.

The proof of the statement from (b) is pursued in a similar way.

Proof of Theorems 3.1 and 3.2. For a given 'trial' subspace U define the vectors $\bar{v}_i$, $i = 1,\ldots,p$ [1]:

$$\bar{v}_i = \operatorname{span}\{u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_p\}^{\perp} \cap U$$

and the subspaces $\bar{V}_j$ and $\bar{W}_{p-j+1}$, $j = 1,\ldots,p$:

$$\bar{V}_j = \operatorname{span}\{\bar{v}_1,\ldots,\bar{v}_j\}$$

$$\bar{W}_{p-j+1} = \operatorname{span}\{\bar{v}_j,\ldots,\bar{v}_p\}.$$

Lemma 4.1. If $\operatorname{tg}\Phi < \infty$, then $\dim \bar{v}_i = 1$, $i = 1,\ldots,p$.

Corollary 4.1. If $\operatorname{tg}\Phi < \infty$, then $\dim \bar{V}_j = j$, $\dim \bar{W}_{p-j+1} = p-j+1$.

Proof of Lemma 4.1. The assumption that $\dim \bar{v}_i = 0$, $i = 1,\ldots,p$, leads to the contradiction with the dimension reasons; the assumption that $\dim \bar{v}_i > 1$ means that in the subspace $\bar{v}_i$ and, hence, in U as well there exists a vector from $\operatorname{span}\{u_{p+1},\ldots,u_{\max}\}$ orthogonal to U_p and this is impossible because $\operatorname{tg}\Phi < \infty$.

Corollary 4.1 is obvious since the vectors $\bar{v}_i$ are linearly independent.

Lemma 4.2. The functional $\lambda^Q(u)$, $u \in U\backslash 0$, where Q is taken from Theorems 3.1 and 3.2 satisfies the variational principle of the Courant–Fischer type:

$$\lambda_j^Q = \min_{V_j \subset H,\, \dim V_j = j} \ \max_{u \in V_j \backslash 0} \lambda^Q(u) = \min_{W_{p-j+1} \subset H,\, \dim W_{p-j+1} = p-j+1} \ \max_{u \in W_{p-j+1} \backslash 0} \lambda^Q(u).$$

Proof. The statement of the lemma is implied by the fact that the denominator of the functional $\lambda^Q(u)$ is positive for all $u \in U\backslash 0$ for the functions from Theorem 3.1 and, by virtue of constraint (3.2), for the functions $Q(\lambda)$ from Theorem 3.2.

Corollary 4.2. The following inequalities are valid:

$$\lambda_j^Q \leqslant \max_{u \in \bar{V}_j \backslash 0} \lambda^Q(u)$$

$$\lambda_j^Q \leqslant \min_{u \in \bar{W}_{p-j+1} \backslash 0} \lambda^Q(u).$$

Let us fix a number $j \in \{1,\ldots,p\}$. Denote by P the orthogonal projector onto U_p, by v_j the vector from $\bar{V}_j\backslash 0$ on which $\max \lambda^Q(u)$, $u \in \bar{V}_j$, is attained and by w_j the vector from $\bar{W}_{p-j+1}\backslash 0$ on which $\min \lambda^Q(u)$, $u \in \bar{W}_{p-j+1}$, is attained. The definitions of $\bar{V}_j$ and $\bar{W}_{p-j+1}$ directly imply the inequalities

$$\lambda_1 \leqslant \lambda^Q(Pv_j) \leqslant \lambda_j \tag{4.3}$$

and

$$\lambda_j \leqslant \lambda^Q(Pw_j) \leqslant \lambda_p. \tag{4.4}$$

In the non-trivial case, we have $v_j - Pv_j = P^\perp v_j \neq 0$ and $w_j - Pw_j = P^\perp w_j \neq 0$. Then if $\operatorname{sign} Q(\lambda) = \text{const} \neq 0$ for $\lambda \in [\underline{\lambda}, \tilde{\lambda}]$, then the following inequalities are obvious:

$$\lambda_{p+1} \leqslant \lambda^Q(P^\perp v_j) \leqslant \lambda_{\max} \tag{4.5}$$

$$\lambda_{p+1} \leqslant \lambda^Q(P^\perp w_j) \leqslant \lambda_{\max}. \tag{4.6}$$

Let us prove the left-hand side of inequality (3.1) (recall that $Q(\lambda) \geqslant 0$ for $\lambda \in [\underline{\lambda}, \tilde{\lambda}]$). The following expression is valid:

$$\lambda_j^Q - \lambda_j \geqslant \lambda^Q(w_j) - \lambda_j$$

$$= \frac{(\lambda^Q(Pw_j) - \lambda_j)(Q(A)Pw_j, Pw_j) + (\lambda^Q(P^\perp w_j) - \lambda_j)(Q(A)P^\perp w_j, P^\perp w_j)}{(Q(A)w_j, w_j)} \geqslant 0.$$

Prove the error estimate from (3.1):

$$\frac{\lambda_j^Q - \lambda_j}{\lambda_{\max} - \lambda_j^Q} \leqslant \frac{\lambda^Q(v_j) - \lambda_j}{\lambda_{\max} - \lambda^Q(v_j)} \leqslant \frac{\lambda^Q(v_j) - \lambda^Q(Pv_j)}{\lambda^Q(P^\perp v_j) - \lambda^Q(v_j)} \equiv J$$

where the first inequality is valid due to the monotonic increase of the function $(x - \lambda_j)/(\lambda_{\max} - x)$ in $x \in [\lambda_j, \lambda_{\max}]$, and the second inequality is valid due to relations (4.3) and (4.5). And since

$$\lambda^Q(v_j) = \frac{(AQ(A)Pv_j, Pv_j) + (AQ(A)P^\perp v_j, P^\perp v_j)}{(Q(A)Pv_j, Pv_j) + (Q(A)P^\perp v_j, P^\perp v_j)}$$

then

$$J = \frac{(Q(A)P^\perp v_j, P^\perp v_j)}{(Q(A)Pv_j, Pv_j)} \leqslant \delta_j \operatorname{tg}^2 \angle(v_j; u_j) \leqslant \delta_j \operatorname{tg}^2 \Phi.$$

Let us prove the left-hand side of inequality (3.3) (recall that $Q(\lambda) \leqslant 0$ for $\lambda \in [\underline{\lambda}, \tilde{\lambda}]$). The following expression is valid:

$$\lambda_j - \lambda_j^Q \geqslant \lambda_j - \lambda^Q(v_j)$$

$$= \frac{(\lambda_j - \lambda^Q(Pv_j))(Q(A)Pv_j, Pv_j) + (\lambda^Q(P^\perp v_j) - \lambda_j)(|Q(A)|P^\perp v_j, P^\perp v_j)}{(Q(A)v_j, v_j)} \geqslant 0.$$

Prove the error estimate from (3.3):

$$\frac{\lambda_j - \lambda_j^Q}{\lambda_{\max} - \lambda_j^Q} \leqslant \frac{\lambda_j - \lambda^Q(w_j)}{\lambda_{\max} - \lambda^Q(w_j)} \leqslant \frac{\lambda^Q(Pw_j) - \lambda^Q(w_j)}{\lambda^Q(P^\perp w_j) - \lambda^Q(w_j)} = \frac{(|Q(A)|P^\perp w_j, P^\perp w_j)}{(Q(A)Pw_j, Pw_j)} \leqslant \sigma_j \operatorname{tg}^2 \Phi$$

where the first inequality is valid due to the fact that the function $(\lambda_j - x)/(\lambda_{\max} - x)$ decreases in $x \leqslant \lambda_j$, and the second inequality is valid due to relations (4.4) and (4.6).

This completes the proof of Theorems 3.1 and 3.2.

The proof of Theorem 3.3 (similar to that of Theorem 3.1) is given, for example, in [4].

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