Fictitious domain methods and computation of homogenized properties of composites with a periodic structure of essentially different components

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1 Introduction

There are many important physical problems that can be modeled using partial differential equations (PDE) with discontinuous coefficients. Drops of the coefficients may be very large. We consider one class of those problems concerning composites of essentially different components.

Composites or composite materials are said to be media with a large number of non homogeneous inclusions with small sizes at least along one direction. Stationary states of that media is described by elliptic PDE with highly oscillated coefficients. The homogenization is said to be a process of finding such homogeneous coefficients that a solution of original PDE can be approximated by the solution of the same PDE, but with these new homogeneous coefficients instead of old highly oscillated coefficients. Computation of homogenized coefficients of composites with periodic structure reduces to solving a series of periodic boundary value problem for the original PDE in the domain, called cell, of periodicity.[1]

For composites of essentially different components the coefficients of the PDE in the domain of periodicity are discontinuous and have big jumps.

Another important source of PDE with jumps of coefficients is Fictitious Domain Method, e.g.[2−8] In this method the original boundary value problem for PDE is changed for a new boundary value problem in a domain that covers the original one. In the new

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fictitious part of the domain the coefficients of PDE are chosen about zero for Neumann original boundary conditions or very large for Dirichlet original boundary conditions, such that a solution of the new problem approximates or, even, coincides with the desired solution. Therefore, the fictitious domain methods makes possible to "improve" the shape of the original boundary, but leads to a PDE with very large jumps of coefficients.

We consider here a nonstandard variant of the fictitious domain method, namely, with periodic boundary conditions on the fictitious boundary. For this variant the problem produced by the fictitious domain method looks exactly like a problem produced by the homogenization procedure we were talking about, i.e., that is a periodic boundary value problem in a cube for elliptic PDE with discontinuous coefficients that can achieve, in limit cases, zero or infinitely large values.

There are several different difficulties associated with numerical solution of those problems. In the present paper we consider only one of them, concerned with the rate of convergence of preconditioned iterative methods. It is well known, that, typically, the larger is the jumps of coefficient, the slower is the convergence. Our main goal is to prove, that with a special initial guess the rate of convergence does not depend on the value of the jumps. We do not touch here the mesh approximations of our PDE, but we can expect that for some natural types of approximations all our results still hold and the rate of convergence does not depend of the mesh size parameter as well.

The central idea of the proof is that with the special initial guess errors of all iterative steps belong to a subspace and in this subspace our problem is well posed independently of the jumps of coefficients, cf.[4]

We also consider some related questions, e.g., the dependence of the solution of small coefficients in a subdomain, cf.[9]

Iterative methods with the same properties were developed earlier for the mixed formulation of PDE.[10,11]

Our investigations base on extension theorems and the Korn inequality.[12,13]

We mention our preliminary publications,[14,15] and further analogous results.[16−19]

The present paper is based on its variant in Russian.[20]

Finally, we note, that through the text we tried to simplify the formulations, e.g., for a factor space we often use an element of a factor class instead of the factor class, without special comments we write equalities that must be viewed in appropriate functional spaces, not in classical sense, etc. However, unique proper rigorous formulations are always possible.

1.1

We consider in the space $\mathbf{R}^s$, typically $s = 2$ or 3, a problem of homogenization of a composite material with the periodic structure be specified by the unit cube $\Pi = (0,1)^s \subset \mathbf{R}^s$, the domain of periodicity. The computation of averaged, or effective, characteristics of a composite material of the periodic structure reduces to the solution of s periodic boundary value problems with special form of right hand sides:[2]

$$\frac{\partial}{\partial \xi_i}\left(A_{ij}\frac{\partial (N_k+\xi_k I)}{\partial \xi_j}\right) = 0,\ k = 1,\ldots,s, \quad (1)$$

where ξ with various indices are independent Cartesian variables in $\mathbf{R}^s$, $\xi = (\xi_1, \dots, \xi_s)^T$, $A_{ij} = A_{ij}(\xi)$ are $m \times m$ periodical matrices of composite coefficients subject to averaging, I is the identity matrix, and the $m \times m$ matrix solution $N_k = N_k(\xi)$ must be periodic also.

Here and below, summation for the repeated indices i and, or, j from 1 to s is understood.

The average coefficients are founded by

$$\hat{A}_{i_1 i_2} = \int_{\Pi} A_{i_1 j}(\xi) \frac{\partial (N_{i_2}(\xi) + \xi_{i_2} I)}{\partial \xi_j} d\,\Pi,\ i_1,\ i_2 = 1, \dots,\ s.$$

In the present paper we consider two basic equations, the diffusion and the elasticity, simultaneously. For the diffusion equation $A_{ij}(\xi)$ are scalar functions, as well as $N_k(\xi)$, and I is the scalar unity. For the elasticity case $A_{ij}(\xi)$ are $s \times s$ matrix functions, as well as $N_k(\xi)$, and I is the $s \times s$ unity matrix.

To unite these cases in one we define a natural parameter m and will treat $A_{ij}(\xi)$ and $N_k(\xi)$ as $m \times m$ matrix functions and I as the $m \times m$ matrix unity.

For the diffusion equation we set $m = 1$.

The case $m > 1$ corresponds to an elliptic system.

We convert an elliptic system with $m = s$ to the elasticity equation by special requirements, that will be formulated in an explicit form every time we need them.

1.2

We denote by A the tensor of the coefficients $A_{ij} = \left(a_{ij}^{kl}\right)$ and suppose the tensor A to be symmetric, i.e.,

$$A_{ij} = A_{ij}^T \text{ or } a_{ij}^{kl} = a_{ji}^{lk}.$$

For the elasticity case we need the following addition conditions,

$$a_{ij}^{kl} = a_{kj}^{il} = a_{il}^{kj} = a_{ji}^{lk}.$$

The system of equations (1) breaks up into m independent systems of partial differential equations for the columns of the matrices N_k of the form

$$\frac{\partial}{\partial \xi_i} \left(A_{ij} \frac{\partial \mathbf{u}}{\partial \xi_j} - \mathbf{f}_i \right) = 0, \tag{2}$$

where the solution $\mathbf{u} = \mathbf{u}(\xi)$ is an m component periodic vector function, and the “right hand sides” — periodic vector functions $\mathbf{f}_i,\ i = 1, \dots, s$, are described by the following rule: every $\mathbf{f}_i$ is the k -th column, $k = 1, \dots,\ m$ of the matrix A_{ij}.

We note, however, that we will not use this restriction and consider equations (2) under general conditions for $\mathbf{f}_i$.

1.3

We define the torus $\mathbf{T}$, multiplication of s unit circles, by the standard identification of the opposite sides of the periodicity domain Π and, further, consider periodic functions in $\mathbf{R}^s$ as functions defined on the torus $\mathbf{T}$, i.e., the vector ξ of independent variables of Cartesian coordinates is now taken on the torus,

$$\xi = (\xi_1, \ldots, \xi_s)^T \in \mathbf{T}. \tag{3}$$

It is conventional to consider the following generalized formulation of the problem (2).

Let all elements of the matrices $A_{ij}(\xi)$ be elements of the space $L_\infty(\mathbf{T})$ and all $\mathbf{f}_i \in \mathbf{L}_2(\mathbf{T}) = \{L_2(\mathbf{T})\}^m$. We define the Hilbert space $\mathbf{H}$ as the factor space $\mathbf{H} = \{W_2^1(\mathbf{T})\}^m/\mathbf{R}^m$ with the scalar product

$$\Lambda(\star,\star) = \left\langle \left(\frac{\partial \star}{\partial \xi_i}, \frac{\partial \star}{\partial \xi_i} \right) \right\rangle,$$

where $\langle\star\rangle$ denotes the (Lebesgue) integral of $\star$ over the torus and $(\star,\star)$ is the natural scalar product of vectors of $\mathbf{R}^m$.

The generalized solution of the problem (2) on the torus $\mathbf{T}$ is said to be such function $\mathbf{v} \in \mathbf{H}_E$ that

$$\Lambda_A(\mathbf{u},\mathbf{v}) = \left\langle \left(\mathbf{f}_i, \frac{\partial \mathbf{v}}{\partial \xi_i} \right) \right\rangle, \; \Lambda_A(\mathbf{u},\mathbf{v}) \stackrel{def}{=} \left\langle \left(A_{ij}(\xi) \frac{\partial \mathbf{u}}{\partial \xi_j}, \frac{\partial \mathbf{v}}{\partial \xi_i} \right) \right\rangle, \; \mathbf{v} \in \mathbf{H}. \tag{4}$$

We note that with (piecewise) smooth coefficients A_{ij} and right hand sides $\mathbf{f}_i$ the solution $\mathbf{u}$ is (piecewise) smooth as well, fulfills (2) in a classical sense in domains of smoothness, and the following standard conditions

$$[\mathbf{u}] = \left[\left(A_{ij} \frac{\partial \mathbf{u}}{\partial \xi_j} - \mathbf{f}_i \right)_{\mathbf{n}} \right] = 0$$

on surfaces of discontinuity.

We note also, that an arbitrary generalized solution of (2) fulfills (2) in the sense of distributions of $\{W_2^{-1}(\mathbf{T})\}^m$.

1.4

Usually, the matrix functions $A_{ij}(\xi) = \left(a_{ij}^{kl}(\xi) \right)$ are subject to the following conditions

$$0 < \underline{a} \le \frac{\sum a_{ij}^{kl}(\xi) \eta_i^k \eta_j^l}{\sum \eta_i^k \eta_j^l} \le \bar{a} < \infty, \; \xi \in \mathbf{T}, \text{ for the elasticity case } \eta_i^k = \eta_k^i, \tag{5}$$

with constants $\underline{a}$ and $\bar{a}$ independent of ξ.

Here and below, summation for the repeated indices k and, or, l from 1 to m is understood. We also recall the rule that summation for the repeated indices i and, or, j from 1 to s is understood. We will put the signs $\sum$ when two independent summations are acquired in a single term.

With conditions (5) the problem (4) is well posed, i.e., there exists a unique solution $\mathbf{u} \in \mathbf{H}$ and

$$\Lambda(\mathbf{u},\mathbf{u}) \le const\langle(\mathbf{f}_i,\mathbf{f}_i)\rangle.$$

We consider the following iterative method

$$\Lambda\left(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\tau},\mathbf{v}\right)+\Lambda_A(\mathbf{u}^n,\mathbf{v})-\left\langle\left(\mathbf{f}_i,\frac{\partial\mathbf{v}}{\partial\xi_i}\right)\right\rangle=0,\ \mathbf{v}\in\mathbf{H},\ n=0,1,\ldots \tag{6}$$

with an arbitrary initial guess $\mathbf{u}^0 \in \mathbf{H}$.

It is easy to see, that with an appropriate $\tau > 0$ iteration approximations $\mathbf{u}^n$ converge to a solution of the problem (4) in $\mathbf{H}$ with the rate of a geometric progression whose convergence factor can be bounded above by a quantity depending only on $\underline{a}/\overline{a}$.

One has to solve periodic boundary value problems for Poisson equation to implement the iterative method (6). Effective numerical methods for such problems are well known, e.g.[3]

1.5

If the ratio $\underline{a}/\overline{a}$ is small due to anisotropy of the material, the following generalization is sometimes justified.

Suppose the $m\times m$ matrices $E_{ij}=\left(e_{ij}^{kl}\right)$, independent of ξ, satisfy the conditions

$$e_{ij}^{kl} = e_{ji}^{lk},\ e_{ij}^{kl}\eta_i^k\eta_j^l>0 \text{ with } \eta\neq 0; \tag{7}$$

for the elasticity case $$e_{ij}^{kl} = e_{kj}^{il}=e_{il}^{kj}=e_{ji}^{lk},\ e_{ij}^{kl}\eta_i^k\eta_j^l>0,\ \eta\neq 0,\ \eta_i^k=\eta_k^i.$$

We set

$$\Lambda_E(\star,\star)\overset{def}{=}\left\langle\left(E_{ij}(\xi)\frac{\partial\star}{\partial\xi_j},\frac{\partial\star}{\partial\xi_i}\right)\right\rangle \text{ in } \mathbf{H}.$$

This bilinear form is a new scalar product in the space $\mathbf{H}$ and generates the norm $\sqrt{\Lambda_E(\star,\star)}$ equivalent to the original norm $\sqrt{\Lambda(\star,\star)}$ of the space $\mathbf{H}$. One of the two inequalities of the equivalence is the Korn inequality on torus $\mathbf{T}$.

We denote by $\mathbf{H}_E$ the function space $\mathbf{H}$ with this new scalar product.

We now redefine constants $\underline{a}$ and $\overline{a}$,

$$0<\underline{a}\le\frac{\sum a_{ij}^{kl}(\xi)\eta_i^k\eta_j^l}{\sum e_{ij}^{kl}\eta_i^k\eta_j^l}\le\overline{a}<\infty,\ \xi\in\mathbf{T},\ \text{for the elasticity case } \eta_i^k=\eta_k^i \tag{8}$$

With conditions (8) the problem (4) is also well posed, i.e., there exists a unique solution $\mathbf{u}\in\mathbf{H}_E$ and

$$\Lambda_E(\mathbf{u},\mathbf{u}) \le const\langle(\mathbf{f}_i,\mathbf{f}_i)\rangle.$$

We consider the iterative method with the new bilinear form,

$$\Lambda_E\left(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\tau},\mathbf{v}\right)+\Lambda_A(\mathbf{u}^n,\mathbf{v})=\left\langle\left(\mathbf{f}_i,\frac{\partial\mathbf{v}}{\partial\xi_i}\right)\right\rangle,\ \mathbf{v}\in\mathbf{H},\ n=0,1,\ldots \tag{9}$$

with an arbitrary initial guess $\mathbf{u}^0 \in \mathbf{H}$.

As in the previous Subsection, one can prove, that with an appropriate $\tau > 0$ iteration approximations $\mathbf{u}^n$ converge to a solution of the problem (4) in $\mathbf{H}_E$ with the rate of a geometric progression whose convergence factor can be bounded above by a quantity depending only on $\underline{a}/\overline{a}$.

In particular,

$$\Lambda_E(\varepsilon^n, \varepsilon^n) \leq q^{2n} \Lambda_E(\varepsilon^0, \varepsilon^0), \; q = 1 - \underline{a}/\overline{a}, \; \varepsilon^n = \mathbf{u}^n - \mathbf{u}, \text{ if } \tau = 1/\overline{a}. \tag{10}$$

By choosing E to make the ratio $\underline{a}/\overline{a}$ as close to one as possible, we can improve the convergence estimates as compared with the method (6).

It is also possible to use more complicated and faster methods, e.g., variation methods, like the conjugate gradient method. We further consider methods of (9) type for simplicity only.

At every iteration of (9) it is necessary to solve periodic boundary value problems for elliptic systems of equations, in particular, for diffusion equation, or for the linear elasticity equations. All the coefficients here are constants and Fourier method can be applied even in case of anisotropy. We also note that Lame equations on a torus with constant Lame parameters are equivalent to several Poisson equations.[11,16]

1.6

Composite materials are characterized not so much by anisotropy of its components as by the inhomogeneity occurring in large drops of the coefficients $a_{ij}^{kl}(\xi)$ in corresponding parts of the periodicity cell. In that situation the ratio $\underline{a}/\overline{a}$ is small for every choice of E of Subsection 1.5.

We decompose the periodicity cell on parts of the two following types. In the "main part" we suppose coefficients $a_{ij}^{kl}(\xi)$ to be of order of one. In the "inclusions" these coefficient may be very small, e.g., "like a cavity" inclusion, or very large, e.g., "almost rigid" inclusion. We also consider limit cases of cavities and rigid inclusions. Such inclusions are typical for composite materials.

Another important source of problems with these properties is the Fictitious Domain Method, e.g.[2−8] The cavities correspond to the Neumann type boundary value problem in the main domain. The rigid inclusions lead to the Dirichlet type boundary value problem in the main domain. A peculiarity of our approach for the fictitious domain method is periodic boundary conditions on the fictitious boundary.

1.7

Let $\mathcal{D} \subset \mathbf{T}$ be itself a Lipschitz domain, i.e., connected opened set with a Lipschitz boundary, or let it consists of a finite number of Lipschitz domains $\mathcal{D}_p$ with nonintersecting closures.

We set $\mathcal{D}^\perp = \mathbf{T} \setminus \mathcal{D}$ and denote by $\mathcal{D}_q^\perp$ connected components of $\mathcal{D}^\perp$, if it is not connected itself.

We take into account that there are a finite number of the subdomains $\mathcal{D}_q^\perp$ and every of them is a Lipschitz domain.

1.8

We suppose the following properties of the coefficients $a_{ij}^{kl}(\xi)$ in the main part $\mathcal{D}$ of the torus $\mathbf{T}$,

$$0 < \underline{a}_{\mathcal{D}} \leq \frac{\sum a_{ij}^{kl}(\xi)\eta_i^k\eta_j^l}{\sum e_{ij}^{kl}\eta_i^k\eta_j^l} \leq \bar{a}_{\mathcal{D}} < \infty,\ \xi \in \mathcal{D}, \tag{11}$$

$$\text{with } \eta_i^k = \eta_k^i \text{ for the elasticity case, } \underline{a}_{\mathcal{D}} \leq 1 \leq \bar{a}_{\mathcal{D}}.$$

We consider the case

$$A_{ij}(\xi) = \omega E_{ij},\ \xi \in \mathcal{D}^{\perp},\ 0 \leq \omega \leq 1 \tag{12}$$

with $\omega = 0$ in Section 2, that corresponds to cavities, and with $\omega > 0$ in Section 3.

The multiparameteric case

$$A_{ij}(\xi) = \omega_q E_{ij},\ \xi \in \mathcal{D}_q^{\perp},\ 0 \leq \omega_q \leq 1 \tag{13}$$

is treated in Section 4.

We do not consider in the present paper the problem (4) with conditions (11) and

$$A_{ij}(\xi) = \omega E_{ij},\ \xi \in \mathcal{D}^{\perp},\ 1 \leq \omega \leq +\infty, \tag{14}$$

that corresponds to rigid inclusions for $\omega = +\infty$, see, e.g.[10,11,14]

We do not cover yet the multiparameteric case

$$A_{ij}(\xi) = \omega_q E_{ij},\ \xi \in \mathcal{D}_q^{\perp},\ 1 \leq \omega_q \leq +\infty, \tag{15}$$

however, we expect that it is possible.

We also mention the conditions

$$\begin{gathered} 0 < \omega\underline{a}_{\mathcal{D}} \leq \frac{\sum a_{ij}^{kl}(\xi)\eta_i^k\eta_j^l}{\sum e_{ij}^{kl}\eta_i^k\eta_j^l} \leq \omega\bar{a}_{\mathcal{D}} < \infty,\ \xi \in \mathcal{D}, \\ \text{with } \eta_i^k = \eta_k^i \text{ for the elasticity case, } \underline{a}_{\mathcal{D}} \leq 1 \leq \bar{a}_{\mathcal{D}}, \\ A_{ij}(\xi) = E_{ij},\ \xi \in \mathcal{D}^{\perp}, \end{gathered} \tag{16}$$

that become conditions (11), (12) or (11), (14) after dividing by ω.

We note that our further results still hold for matrices E_{ij} dependent of ξ as well, if the condition of (5) is fulfilled for E_{ij} instead of the inequality of (7). However, a practical implementation of iterative method (9) becomes harder, inner iterative procedures should be involved. We do not consider that possibility in the present paper.

2 Fictitious gradients method. Perforated composites.

2.1

We consider a problem (4):

$$\Lambda_A(\mathbf{u},\mathbf{v}) = \langle(\mathbf{f}_i, \partial\mathbf{v}/\partial\xi_i)\rangle,\ \mathbf{v} \in \mathbf{H}, \tag{17}$$

in conditions (11), (12) with $\omega = 0$, i.e.

$$0 < \underline{a}_{\mathcal{D}} \leq \frac{\sum a_{ij}^{kl}(\xi)\eta_i^k\eta_j^l}{\sum e_{ij}^{kl}\eta_i^k\eta_j^l} \leq \bar{a}_{\mathcal{D}} < \infty, \; \xi \in \mathcal{D}, \; \eta_i^k = \eta_k^i \text{ for the elasticity case,} \tag{18}$$

$$A_{ij}(\xi) = 0, \; \xi \in \mathcal{D}^{\perp}.$$

Conditions (18) correspond to porous media with the subdomain $\mathcal{D}^{\perp}$ constitutes a set of cavities. Such media can be treated as a perforated composite material.

We prove in the Subsection 2.3, that the problem (17) with (18) is the problem of the Fictitious Domain Method applied to the Neumann boundary value problem in $\mathcal{D}$.

Our approach to the solution of (17) here has much in common with the familiar method.[2] The main difference is that we suggest periodic boundary conditions on the fictitious boundary, and the traditional choice is the Dirichlet or Neumann boundary conditions. This leads to some difficulties of the theory, but leads also to advantages of a practical implementation, because periodic boundary value problem for elasticity equations can be solved efficiently.

2.2

We denote the tensor of the coefficients $A \equiv A^{\omega}$ by A^0 to underline that $\omega = 0$ in this Section. Bilinear form $\Lambda_{A^0}(\star,\star)$ has a kernel

$$\mathbf{Ker} = \left\{ \mathbf{w} \in \mathbf{H} : \left\langle \left(A_{ij}^0 \frac{\partial \mathbf{w}}{\partial \xi_j}, \frac{\partial \mathbf{v}}{\partial \xi_i} \right) \right\rangle = 0, \; \mathbf{v} \in \mathbf{H} \right\}.$$

That consists of vector functions $\mathbf{w} \in \mathbf{H}$, which must obey the equality $\mathbf{w} = \mathbf{c}_p$ with a constant vector $\mathbf{c}_p$ in each component of connectedness $\mathcal{D}_p$. For the elasticity case $\mathbf{w} = \mathbf{c}_p + C_p\xi$ in $\mathcal{D}_p$ with some vector $\mathbf{c}_p$ and an $s \times s$ matrix $C_p = -C_p^T$, both independent of ξ

We have to comment the use of the function $C_p\xi$ in $\mathcal{D}_p$. If $s \times s$ matrix $C = -C^T \neq 0$ then the function $C\xi$ with $\xi \in \mathbf{R}^s$ is not periodic. Therefore, when passing from $\mathbf{R}^s$ to the torus $\mathbf{T}$ this function rearranges to a multiplace function with single-valued branches differ from each other to constant vectors with integer components. Evidently, a restriction $C\xi\,|_{\mathcal{D}}$ of the function $C\xi$ on an open subset $\mathcal{D} \subset \mathbf{T}$ is multi-valued also. We would like to consider the restriction $C\xi\,|_{\mathcal{D}}$ as a restriction of a function in the space $\mathbf{H}$, and only continuous single-value branch of the function $C\xi\,|_{\mathcal{D}}$ could play that role.

This motivates the next

Definition 2.1 *A "continuous restriction" $C\xi\,|_{\mathcal{D}}$ is called a continuous single-value branch of the multiplace function $C\xi\,|_{\mathcal{D}}$ in $\mathcal{D}$, if such a branch exists for the given matrix C and the given open subset $\mathcal{D} \subset \mathbf{T}$.*

We note, using this definition, that subspace **Ker** for the elasticity case consists of vector functions $\mathbf{w} \in \mathbf{H}$, which must obey the equality $\mathbf{w} = \mathbf{c}_p + C_p\xi$ in $\mathcal{D}_p$ with some vector $\mathbf{c}_p$ and an $s \times s$ matrix $C_p = -C_p^T$, both independent of ξ and restriction $C\xi\,|_{\mathcal{D}}$ must be continuous in each component of connectedness $\mathcal{D}_p$.

The condition of continuity of the restriction $C\xi\,|_{\mathcal{D}}$ is fulfilled for all $s \times s$ matrices $C_p = -C_p^T$, independent of ξ, if the domain $\mathcal{D}_p$ not *surrounds* the torus $\mathbf{T}$.

Definition 2.2 *An open subset $\mathcal{D} \subset \mathbf{T}$ is said to not "surround" the torus $\mathbf{T}$, if all components of connectedness of the image of $\mathcal{D}$, when torus $\mathbf{T}$ is periodically extended on $\mathbf{R}^s$, are bounded in $\mathbf{R}^s$.*

If $\mathcal{D}_p$ surrounds the torus $\mathbf{T}$, then the condition of continuity of the restriction $C\xi\,|_{\mathcal{D}}$ leads to additional constraints on the matrix $C_p = -C_p^T$, — it must have a kernel. For example, if $\mathcal{D}_p = \mathbf{T}$ or $\mathcal{D}_p = \{\xi \in \mathbf{T} : 0 < \xi_1 < 1/2\}$, then $C_p = 0$ necessarily. In general case of surrounding the constrains seem to be like this: some linear combinations of columns of C_p with integer coefficients equal to zero, and values of the integer coefficients are defined by directions of surrounding.

Let $\mathbf{v} \in \mathbf{Ker}$ in (17), then

$$\langle (\mathbf{f}_i, \partial \mathbf{v}/\partial \xi_i) \rangle = 0, \ \mathbf{v} \in \mathbf{Ker}, \tag{19}$$

that is necessary condition on right hand sides $\mathbf{f}_i$, for existence of a solution of the problem (17) in the space $\mathbf{H}$. We will prove in Theorem 2.1 that it is a sufficient condition as well.

If the problem (17) with (18) has a solution $\mathbf{u} \in \mathbf{H}$ for given $\mathbf{f}_i$, then any function of the type $\mathbf{u} + \mathbf{w}, \ \mathbf{w} \in \mathbf{Ker}$ is a solution again. We will choose from that solutions the *normal* one.

Definition 2.3 *The "normal" solution is called a solution with minimal norm in $\mathbf{H}_E$ among all solutions for given $\mathbf{f}_i$.*

Lemma 2.1 *Let the problem (17) with (18) has a solution in $\mathbf{H}$ for given $\mathbf{f}_i$.*

Then the normal solution exists, is unique, and lies in the subspace

$$\mathbf{Im} = \{\mathbf{w} \in \mathbf{H} : \ \Lambda_E(\mathbf{w}, \mathbf{v}) = 0, \ \mathbf{v} \in \mathbf{Ker}\}.$$

The proof of Lemma 2.1 bases on the decomposition $\mathbf{H}_E = \mathbf{Ker} \oplus \mathbf{Im}$ and is a standard one.

Theorem 2.1 *Suppose conditions (18) and (19) are met. Let matrices E_{ij} satisfy requirements of Subsection 1.5 and a set $\mathcal{D}$ compiles with requirements of Subsection 1.7.*

Then the problem (17) has a unique normal solution $\mathbf{u} \in \mathbf{H}$ and

$$\Lambda_E(\mathbf{u}, \mathbf{u}) \leq const \ \langle (\mathbf{f}_i, \mathbf{f}_i) \rangle. \tag{20}$$

In the next theorem we state, that the iterative method (9) can be applied for effective solution of the problem (17) with (18) as well as of the problem (4) with (8), but the special initial guess must be chosen.

Theorem 2.2 *Let conditions of Theorem 2.1 be satisfied. We consider the iterative method (9):*

$$\Lambda_E\left(\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\tau}, \mathbf{v}\right) + \Lambda_{A^0}(\mathbf{u}^n, \mathbf{v}) = \left\langle \left(\mathbf{f}_i, \frac{\partial \mathbf{v}}{\partial \xi_i}\right) \right\rangle, \ \mathbf{v} \in \mathbf{H}, \ n = 0, 1, \ldots \tag{21}$$

with the initial guess $\mathbf{u}^0$, *a solution of*

$$\Lambda_E(\mathbf{u}^0, \mathbf{v}) = \langle(\mathbf{g}_i, \partial\mathbf{v}/\partial\xi_i)\rangle, \ \mathbf{v} \in \mathbf{H}, \tag{22}$$

where $\mathbf{g}_i \in \mathbf{L_2}(\mathbf{T})$ *are arbitrary functions such that*

$$\langle(\mathbf{g}_i, \partial\mathbf{v}/\partial\xi_i)\rangle = 0, \ \mathbf{v} \in \mathbf{Ker},$$

cf. with (19), for example, $\mathbf{g}_i \equiv \mathbf{0}$.

For an appropriate $\tau > 0$ *iteration approximations* $\mathbf{u}^n$ *converge to the normal solution of the problem (17) in* $\mathbf{H}_E$ *with the rate of a geometric progression whose convergence factor can be bounded above by a quantity depending only on* $\kappa\underline{a}_{\mathcal{D}}/\overline{a}_{\mathcal{D}}$.

In particular,

$$\Lambda_E(\varepsilon^n, \varepsilon^n) \le q^{2n}\Lambda_E(\varepsilon^0, \varepsilon^0), \ q = 1 - \kappa\underline{a}_{\mathcal{D}}/\overline{a}_{\mathcal{D}}, \ \text{if } \tau = 1/\overline{a}_{\mathcal{D}}. \tag{23}$$

Here $\kappa > 0$ *is the constant of the following proposition of extension in* $\mathbf{H}_E$ *from* $\mathcal{D}$ *to* $\mathbf{T}$:

Proposition 2.1 *For any function* $\mathbf{v} \in \mathbf{H}$ *there exists a function* $\mathbf{w} \in \mathbf{H}$ *such that*

$$\int_{\mathcal{D}} \left(E_{ij}\frac{\partial\mathbf{v}}{\partial\xi_j}, \frac{\partial\mathbf{v}}{\partial\xi_i}\right) d\mathcal{D} \ge \kappa \int_{\mathbf{T}} \left(E_{ij}\frac{\partial\mathbf{w}}{\partial\xi_j}, \frac{\partial\mathbf{w}}{\partial\xi_i}\right) d\mathbf{T}, \mathbf{w} - \mathbf{v} \in \mathbf{Ker}.$$

This proposition plays an important role. The same statement is not true, in general, if the boundary of $\mathcal{D}$ is not Lipschitz. We will prove the proposition at the last Section.

We now consider a special case of constant coefficients $A_{ij}(\xi)$ in $\mathcal{D}$.

Theorem 2.3 *Let conditions of Theorem 2.2 be satisfied. We suppose in addition that*

$$A^0_{ij}(\xi) = E_{ij}, \ \xi \in \mathcal{D} \tag{24}$$

and take $\mathbf{g}_i = \mathbf{f}_i$ *in (22) to find an initial guess.*

Then

$$supp\ \mathbf{r}^n = supp \frac{\partial}{\partial\xi_i}\left(E_{ij}\frac{\partial(\mathbf{u}^{n+1} - \mathbf{u}^n)}{\partial\xi_j}\right) \subseteq \partial\mathcal{D}, \tag{25}$$

where

$$\mathbf{r}^n = \frac{\partial}{\partial\xi_i}\left(A_{ij}\frac{\partial\mathbf{u}^n}{\partial\xi_j} - \mathbf{f}_i\right) \in \{W_2^{-1}(\mathbf{T})\}^m$$

are residuals.

Proofs of all these theorems base on the next key statement.

Lemma 2.2 *Let conditions of Theorem 2.2 be satisfied.*

Then

1. *Initial guess of (22)* $\mathbf{u}^0 \in \mathbf{Im}$.
2. *In the subspace* $\mathbf{Im}$

$$0 < \kappa\underline{a}_{\mathcal{D}} \le \frac{\Lambda_A(\mathbf{v}, \mathbf{v})}{\Lambda_E(\mathbf{v}, \mathbf{v})} \le \overline{a}_{\mathcal{D}} < \infty, \ \mathbf{v} \in \mathbf{Im}. \tag{26}$$

3. *Subspace* $\mathbf{Im}$ *is an invariant subspace of the operator* $L : \mathbf{H} \to \mathbf{H}$ *defined by*

$$\Lambda_E(L\mathbf{w}, \mathbf{v}) = \Lambda_A(\mathbf{w}, \mathbf{v}), \ \mathbf{w}, \ \mathbf{v} \in \mathbf{H}. \tag{27}$$

2.3

We now check that the problem (17) with (18) can be viewed as a problem of the fictitious domain method applied to Neumann boundary value problem in $\mathcal{D}$. To make our consideration more simple we suppose during this Subsection, that all

$$\mathbf{f}_i = \mathbf{f}_i(\xi) = 0,\ \xi \in \mathcal{D}^{\perp}. \tag{28}$$

Then the problem (17) takes the following form

$$\int\limits_{\mathcal{D}} \left(A^0_{ij}(\xi) \frac{\partial \mathbf{u}}{\partial \xi_j}, \frac{\partial \mathbf{v}}{\partial \xi_i} \right) d\mathcal{D} = \int\limits_{\mathcal{D}} \left(\mathbf{f}_i, \frac{\partial \mathbf{v}}{\partial \xi_i} \right) d\mathcal{D},\ \mathbf{v} \in \mathbf{H}. \tag{29}$$

The subset $\mathcal{D} \subset \mathbf{T}$ is either connected by itself, or consists of finite number of connected components $\mathcal{D}_p$ whose closures are not intersected with each other, as it was assumed at Subsection 1.7.

For the first case,

$$\{W^1_2(\mathbf{T})\}^m \mid_{\mathcal{D}} = \{W^1_2(\mathcal{D})\}^m,$$

for the second, the space $\{W^1_2(\mathbf{T})\}^m \mid_{\mathcal{D}}$ is a direct multiplication of spaces $\{W^1_2(\mathcal{D}_p)\}^m$ for all different p. Therefore, the problem (29) is a variation formulation of the next Neumann boundary value problem

$$\frac{\partial}{\partial \xi_i} \left(A^0_{ij} \frac{\partial \mathbf{u}}{\partial \xi_j} - \mathbf{f}_i \right) = 0,\ \xi \in \mathcal{D},\ \left(A^0_{ij} \frac{\partial \mathbf{u}}{\partial \xi_j} - \mathbf{f}_i \right)_{\mathbf{n}} \Bigg|_{\partial \mathcal{D}} = 0. \tag{30}$$

For a nonconnected $\mathcal{D}$ this problem falls into several independent analogous problems in every domain $\mathcal{D}_p$.

In other words, the solution of the problem (17) with (18) is, at the same time, a generalized solution in $\mathcal{D}$ of the Neumann boundary value problem(s) (30). As we have already noted, an arbitrary function $\mathbf{w} \in \mathbf{Ker}$ can be added to a solution of the problem (17) with (18) and the sum also will be a solution. For the elasticity case all kinds of functions $\mathbf{w} \in \mathbf{Ker}$ in $\mathcal{D}_p$ describe all possible shifts and rotations of $\mathcal{D}_p$ in $\mathbf{T}$. This corresponds to well known fact of linear elasticity theory, e.g.[12], that shifts and rotations are trivial solutions of the Neumann boundary value problem.

2.4

We now consider conditions for the right hand sides $\mathbf{f}_i$ more carefully.

The Neumann problem (30) has a solution for arbitrary right hand sides $\mathbf{f}_i$, except the case of elasticity. For that case the same statement is true, if $\mathcal{D} = \mathbf{T}$. It is also well known in the elasticity case, e.g.[12], that the necessary and sufficient condition of solvability of (30) is vanishing of the average moment of applied forces, i.e.,

$$F_p - F_p^T = 0, \tag{31}$$

where $s \times s$ matrix F_p is as an average in $\mathcal{D}_p$ of $s \times s$ matrix composed of the right hand sides $\mathbf{f}_i$:

$$F_p = \int\limits_{\mathcal{D}_p} (\mathbf{f}_1\ \dots\ \mathbf{f}_m)\, d\mathcal{D},$$

if $\mathcal{D}_p$ do not surround the torus $\mathbf{T}$.

For our problem (17) with (18) we have stated already in Theorem 2.1, that (19) is the desired condition.

We suppose (28) for simplicity during the rest of this Subsection, and prove that the condition (19) always holds, except the elasticity case. For that case we discover a specific analog of (19) for general domains $\mathcal{D}_p$ and recognize that it is the same as (31), if $\mathcal{D}$ is connected and not surrounds the torus $\mathbf{T}$.

In view of (28) condition (19) is equivalent to a set of independent conditions

$$\int\limits_{\mathcal{D}_p} \left(\mathbf{f}_i, \frac{\partial \mathbf{v}}{\partial \xi_i}\right) d\mathcal{D}_p = 0, \ \mathbf{v} \in \mathbf{Ker}. \tag{32}$$

If $\mathbf{v} \in \mathbf{Ker}$, then $\mathbf{v} = \mathbf{c}_p$ in $\mathcal{D}_p$ with a constant vector $\mathbf{c}_p$, except the elasticity case. Substituting $\mathbf{c}_p$ instead of $\mathbf{v}$ in (32) we conclude, that condition (32) is fulfilled.

The elasticity case is more complicated. We substitute at (31) $\mathbf{v} = \mathbf{c}_p + C_p\xi$ in $\mathcal{D}_p$ with a vector $\mathbf{c}_p$ and an $s \times s$ matrix $C_p = -C_p^T$ both independent of ξ. We differentiate (formally) of ξ_i :

$$\frac{\partial \mathbf{w}}{\partial \xi_i} = \frac{\partial (C_p\xi)}{\partial \xi_i}$$

equals to the i-th column of the matrix C_p. Therefore, condition (32) can be written in the following equivalent algebraic form

$$tr(F_p^T C_p) = 0.$$

Taking into account the equality $C_p = -C_p^T$ and properties of the trace operation, we rewrite

$$tr(\{F_p - F_p^T\}C_p) = 0. \tag{33}$$

Here F_p is an average in $\mathcal{D}_p$ of $s \times s$ matrix composed of the right hand sides $\mathbf{f}_i$ as in (31) and C_p is an arbitrary $s \times s$ matrix $C_p = -C_p^T$ independent of ξ and such that the restriction $C_p\xi\,|_{\mathcal{D}_p}$ is continuous, see Definition 2.1.

A collection of conditions (33) with (28) for all different $\mathcal{D}_p$ is the wanted concrete variant of the solvability condition (19) for the elasticity case. It is by that a necessary and a sufficient condition of $\mathbf{f}_i$ for (generalized) solvability of Neumann boundary value problem (30) for the elasticity case.

We illustrate the condition by several examples:

1. Let $\mathcal{D}_p$ be not surrounding the torus $\mathbf{T}$, see Definition 2.2. For example, let $\mathcal{D}_p = \{\xi : \xi_i\xi_i < 1/4\}$. Then $C_p = -C_p^T$ can be arbitrary. We take $C_p = F_p - F_p^T$ in (33) and are led to (31).

2. An opposite extreme case is when $\mathcal{D}_p$ surrounds the torus $\mathbf{T}$ every which Cartesian direction except, may be, one. For example, let $\mathcal{D}_p = \{\xi : 0 < \xi_1 < 1/2\}$. Then $C_p = 0$, i.e., the domain $\mathcal{D}_p$ cannot be rotated in $\mathbf{T}$. The condition (33) becomes trivially fulfilled for arbitrary $\mathbf{f}_i$.

3. Let $\mathcal{D}_p = \{\xi : 0 < \xi_1, \xi_2 < 1/2\}$, $s > 2$. Then in the matrix C_p there are only two nonvanished elements and they have indexes (1,2) and (2.1). Follow (33), we conclude that

$$\int_{\mathcal{D}_p} (f_{12} - f_{21}) d\mathcal{D}_p = 0.$$

We note that for the special right hand sides $\mathbf{f}_i$ of (3), which come from the homogenization process, the equalities $f_{ik} = f_{ki}$ are true for the elasticity case. Therefore, the condition (33) is fulfilled independently of a shape of $\mathcal{D}_p$.

2.5

This subsection contains proofs.

Proof of Lemma 2.2.

1. By definition of the subspace **Im** it is sufficient to check the equality

$$\Lambda_E(\mathbf{u}^0, \mathbf{v}) = 0, \ \mathbf{v} \in \mathbf{Ker}.$$

But it readily apparent from the formulae (22) for $\mathbf{u}^0$.

2. (a) *Except the elasticity case.* We first prove, that the right inequality of (26) holds even for an arbitrary $\mathbf{v} \in \mathbf{H}$. Taking into account (18) we have

$$\Lambda_{A^0}(\mathbf{v}, \mathbf{v}) = \int_{\mathcal{D}} \left(A^0_{ij}(\xi) \frac{\partial \mathbf{v}}{\partial \xi_j}, \frac{\partial \mathbf{v}}{\partial \xi_i} \right) d\mathcal{D} \le \bar{a}_{\mathcal{D}} \int_{\mathcal{D}} \left(E_{ij} \frac{\partial \mathbf{v}}{\partial \xi_j}, \frac{\partial \mathbf{v}}{\partial \xi_i} \right) d\mathcal{D}$$

By requirements imposed on matrices E_{ij} in Subsection 1.5, the last integral cannot decrease when the domain $\mathcal{D}$ of integration is changed to $\mathbf{T}$, which concludes the proof of the right inequality.

The first step of checking of the left inequality is the same as above:

$$\Lambda_{A^0}(\mathbf{v}, \mathbf{v}) \ge \underline{a}_{\mathcal{D}} \int_{\mathcal{D}} \left(E_{ij} \frac{\partial \mathbf{v}}{\partial \xi_j}, \frac{\partial \mathbf{v}}{\partial \xi_i} \right) d\mathcal{D}, \ \mathbf{v} \in \mathbf{H}.$$

The second (and the last) step is establishing the following

$$\int_{\mathcal{D}} \left(E_{ij} \frac{\partial \mathbf{v}}{\partial \xi_j}, \frac{\partial \mathbf{v}}{\partial \xi_i} \right) d\mathcal{D} \le \kappa \int_{\mathbf{T}} \left(E_{ij} \frac{\partial \mathbf{v}}{\partial \xi_j}, \frac{\partial \mathbf{v}}{\partial \xi_i} \right) d\mathbf{T}, \ \mathbf{v} \in \mathbf{Im}. \tag{34}$$

For that we use the Proposition of extension for the function $\mathbf{v}$, i.e., there is a function $\mathbf{w} \in \mathbf{H}$, such that

$$\int_{\mathcal{D}} \left(E_{ij} \frac{\partial \mathbf{v}}{\partial \xi_j}, \frac{\partial \mathbf{v}}{\partial \xi_i} \right) d\mathcal{D} \le \kappa \int_{\mathbf{T}} \left(E_{ij} \frac{\partial \mathbf{w}}{\partial \xi_j}, \frac{\partial \mathbf{w}}{\partial \xi_i} \right) d\mathbf{T}, \ \mathbf{v} \in \mathbf{Im}.$$

and $\mathbf{w}-\mathbf{w} \in \mathbf{Ker}$. But our function $\mathbf{v}$ belongs to $\mathbf{Im}$ that is orthogonal complement of $\mathbf{Ker}$ in $\mathbf{H}_E$, therefore

$$\int_{\mathbf{T}} \left(E_{ij} \frac{\partial \mathbf{w}}{\partial \xi_j}, \frac{\partial \mathbf{w}}{\partial \xi_i} \right) d\mathbf{T} = \Lambda_E(\mathbf{w},\mathbf{w}) \geq \Lambda_E(\mathbf{v},\mathbf{v}),\ \mathbf{v} \in \mathbf{Im}.$$

Inequality (34) and, thus, (26) have proved.

(b) *For the elasticity case.* We remind that for the elasticity case indexes $i,\ j,\ k,\ l$ vary from 1 to $s=m$, consider components v_l of the displacement vector $\mathbf{v} \in \mathbf{H},\ \mathbf{v}=(v_1,\ldots,v_s)^T$ and define components

$$\varepsilon_{lj} = \frac{1}{2}\left(\frac{\partial v_l}{\partial \xi_j} + \frac{\partial v_j}{\partial \xi_l} \right)$$

of the symmetric tensor of deformations ε. Using the symmetry of the tensor A of elastic modulus we obtain well known equalities for components σ_{ki} of the stress tensor

$$\sigma_{ki} = a_{ij}^{kl} \varepsilon_{lj} = a_{ij}^{kl} \frac{\partial v_l}{\partial \xi_j}. \tag{35}$$

From (35) it follows that

$$\Lambda_{A^0}(\mathbf{v},\mathbf{v}) = \int_{\mathcal{D}} \left(A_{ij}^0(\xi) \frac{\partial \mathbf{v}}{\partial \xi_j}, \frac{\partial \mathbf{v}}{\partial \xi_i} \right) d\mathcal{D} = \int_{\mathcal{D}} a_{ij}^{kl} \varepsilon_{lj} \varepsilon_{ki} d\mathcal{D}. \tag{36}$$

Taking into account conditions (18) we get

$$\underline{a}_{\mathcal{D}} \leq \frac{\int_{\mathcal{D}} a_{ij}^{kl} \varepsilon_{lj} \varepsilon_{ki} d\mathcal{D}}{\int_{\mathcal{D}} e_{ij}^{kl} \varepsilon_{lj} \varepsilon_{ki} d\mathcal{D}} \leq \bar{a}_{\mathcal{D}}. \tag{37}$$

Tensor $\mathbf{E}$ is symmetric as well as tensor $\mathbf{A}$, therefore, by analogy with (35) and (36), the equalities

$$e_{ij}^{kl} \varepsilon_{lj} = e_{ij}^{kl} \frac{\partial v_l}{\partial \xi_j},$$

$$\int_{\mathcal{D}} e_{ij}^{kl} \varepsilon_{lj} \varepsilon_{ki} d\mathcal{D} = \int_{\mathcal{D}} \left(E_{ij} \frac{\partial \mathbf{v}}{\partial \xi_j}, \frac{\partial \mathbf{v}}{\partial \xi_i} \right) d\mathcal{D}$$

hold. The proof is further the same as the previous one for the case except the elasticity.

3. We shall see, that $L\mathbf{w} \in \mathbf{Im}$ even for an arbitrary function $\mathbf{w} \in \mathbf{H}$. We have to check the equality $\Lambda_E(L\mathbf{w},\mathbf{v}) = 0$ for $\mathbf{v} \in \mathbf{Ker}$. But this equality follows directly from the definition (27) of the operator L, because the subspace $\mathbf{Ker}$ is a kernel of the symmetric bilinear form $\Lambda_{A^0}(\star,\star)$.

Π

We note that the operator $L : \mathbf{H} \to \mathbf{H}$ is selfadjoint in $\mathbf{H}_E$ and inequalities (26) are equivalent to

$$\kappa \underline{a}_{\mathcal{D}} I \leq L \leq \bar{a}_{\mathcal{D}} I \text{ in } \mathbf{Im} \subset \mathbf{H}_E. \tag{38}$$

We conclude that the image $\mathbf{Im}\, L$ of the operator L is closed in $\mathbf{H}$ and $\mathbf{Im}\, L = \mathbf{Im}$ and the subspace $\mathbf{Ker}$ is a kernel of operator L.

Proof of Theorem 2.1. We represent vector $\mathbf{v} \in \mathbf{H}$ of (17) as an orthogonal in $\mathbf{H}_E$ sum,

$$\mathbf{v} = \mathbf{v}_K + \mathbf{v}_I,\ \mathbf{v}_K \in \mathbf{Ker},\ \mathbf{v}_I \in \mathbf{Im},$$

and use the additive property. Terms with $\mathbf{v}_K$ all vanish and the problem (17) takes the following form

$$\Lambda_A(\mathbf{u},\mathbf{v}) = \langle (\mathbf{f}_i, \partial \mathbf{v}/\partial \xi_i) \rangle,\ \mathbf{v} \in \mathbf{Im}. \tag{39}$$

A solution $\mathbf{v}$ here can be found in the subspace $\mathbf{Im}$, as was stated in Lemma 2.1. By Lemma 2.2 the bilinear form $\Lambda_{A^0}(\star,\star)$ in the subspace $\mathbf{Im}$ is $\mathbf{H}_E$ bounded and $\mathbf{H}_E$ coercivity. The linear functional $\langle (\mathbf{f}_i, \partial \mathbf{v}/\partial \xi_i) \rangle$, $\mathbf{v} \in \mathbf{H}$, is $\mathbf{H}_E$ bounded, because

$$|\langle (\mathbf{f}_i, \partial \mathbf{v}/\partial \xi_i) \rangle|^2 \leq \langle (\mathbf{f}_i, \mathbf{f}_i) \rangle \Lambda(\mathbf{u},\mathbf{v}) \leq c\ \langle (\mathbf{f}_i, \mathbf{f}_i) \rangle \Lambda_E(\mathbf{u},\mathbf{v}),\ \mathbf{v} \in \mathbf{H},$$

where for the elasticity case c is a constant of the Korn's inequality for functions in torus $\mathbf{T}$, e.g.[12] Using standard arguments,[21] we conclude, that the problem (39) is correct in $\mathbf{H}_E$ and the constant in (20) equals to the ratio $\frac{c}{\kappa \underline{a}_{\mathcal{D}}}$.

We now prove the statement once more, taking advantage of an operator language.

Equation (17) can be written in equivalent operator form as

$$L\mathbf{u} = \mathbf{w},\ \mathbf{u} \in \mathbf{Im},\ \mathbf{w} \in \mathbf{Im}, \tag{40}$$

where $\mathbf{w}$ is a canonical representation in $\mathbf{H}_E$ of the linear bounded functional $\langle (\mathbf{f}_i, \partial \mathbf{v}/\partial \xi_i) \rangle$, $\mathbf{v} \in \mathbf{H}$. By Lemma 2.2 the operator $L : \mathbf{Im} \to \mathbf{Im}$ has its bounded in $\mathbf{H}_E$ inverse with a norm less or equal to the ratio $\frac{1}{\kappa \underline{a}_{\mathcal{D}}}$. Π

Proof of Theorem 2.2. We first check $\mathbf{u}^n \in \mathbf{Im},\ n = 0,\ 1, \ldots,$ using induction.

We have $\mathbf{u}^0 \in \mathbf{Im}$ by Lemma 2.2.

Let $\mathbf{u}^n \in \mathbf{Im}$. We set $\mathbf{v} \in \mathbf{Ker}$ in (21) and get

$$\Lambda_E(\mathbf{u}^{n+1} - \mathbf{u}^n, \mathbf{v}) = 0,\ \mathbf{v} \in \mathbf{Ker},$$

follow definition of $\mathbf{Ker}$ and condition (19). This means $\mathbf{u}^{n+1} - \mathbf{u}^n \in \mathbf{Im}$ by definition of $\mathbf{Im}$.

We now define the function of error $\varepsilon^n = \mathbf{u}^n - \mathbf{u}$ of the n - th iteration for our iterative process (21), where $\mathbf{u}$ is the normal solution of (17). Lemma 2.1 states, that the normal solution $\mathbf{u} \in \mathbf{Im}$, therefore $\varepsilon^n \in \mathbf{Im}$ as well. It is easy to rewrite the iterative process (21) for the errors $\varepsilon^n \in \mathbf{Im}$ in the subspace $\mathbf{Im}$:

$$\Lambda_E\left(\frac{\varepsilon^{n+1} - \varepsilon^n}{\tau}, \mathbf{v}\right) + \Lambda_{A^0}(\varepsilon^n, \mathbf{v}) = 0,\ \mathbf{v} \in \mathbf{H},\ n = 0, 1, \ldots,\ \varepsilon^0 \in \mathbf{Im}. \tag{41}$$

We also can convert iterations (41) to the following operator form

$$\varepsilon^{n+1} = (I - \tau L)\varepsilon^n,\ n = 0,\ 1, \ldots,\ \varepsilon^0 \in \mathbf{Im}, \tag{42}$$

where operator $I - \tau L$ of iteration step from ε^n to ε^{n+1} is a compression in $\mathbf{Im} \subset \mathbf{H}_E$, i.e.,

$$\|I - \tau L\|_E \leq q < 1 \text{ in } \mathbf{Im}$$

with some q, if τ is an appropriate one. For example, $q = 1 - \kappa \underline{a}_{\mathcal{D}}/\overline{a}_{\mathcal{D}}$, if $\tau = 1/\overline{a}_{\mathcal{D}}$. It follows from (38) directly and completes the proof. Π

Proof of Theorem 2.3. We first note, that

$$\tau \mathbf{r}^n = \frac{\partial}{\partial \xi_i}\left(E_{ij}\frac{\partial(\mathbf{u}^{n+1} - \mathbf{u}^n)}{\partial \xi_j}\right).$$

We now verify that conditions (18) without (24) implies the following inclusion

$$supp \frac{\partial}{\partial \xi_i}\left(E_{ij}\frac{\partial(\mathbf{u}^{n+1} - \mathbf{u}^n)}{\partial \xi_j}\right) \subseteq \overline{\mathcal{D}}, \tag{43}$$

and, moreover, an arbitrary function $\mathbf{w} \in \mathbf{Im}$ can be here instead of the difference $\mathbf{u}^{n+1} - \mathbf{u}^n$. That inclusion means

$$\Lambda_E(\mathbf{w}, \mathbf{v}) = \int_{\mathbf{T}} \left(E_{ij}\frac{\partial \mathbf{w}}{\partial \xi_j}, \frac{\partial \mathbf{v}}{\partial \xi_i}\right) d\mathbf{T} = 0 \tag{44}$$

with a smooth function

$$\mathbf{v} \in \mathbf{H},\ supp\ \mathbf{v} \subseteq \mathbf{T} \setminus \mathcal{D}.$$

But any of such a function $\mathbf{v} \in \mathbf{Ker}$, and desired (44) holds because of $\mathbf{w} \in \mathbf{Im}$.

It remains to find out that condition (24) leads to

$$supp \frac{\partial}{\partial \xi_i}\left(E_{ij}\frac{\partial(\mathbf{u}^{n+1} - \mathbf{u}^n)}{\partial \xi_j}\right) \subseteq \mathbf{T} \setminus \mathcal{D}, \tag{45}$$

If so, then (43) and (45) constitute (25).

We shall prove (45) for a more general case in the next Section, see Proof of Theorem 3.4. Π

3 Composites with inclusions of a soft material

3.1

We consider a problem (4):

$$\Lambda_A(\mathbf{u}, \mathbf{v}) = \langle(\mathbf{f}_i, \partial \mathbf{v}/\partial \xi_i)\rangle,\ \mathbf{v} \in \mathbf{H}, \tag{46}$$

in conditions (11), (12) with $\omega > 0$, i.e.

$$0 < \underline{a}_{\mathcal{D}} \leq \frac{\sum a_{ij}^{kl}(\xi)\eta_i^k\eta_j^l}{\sum e_{ij}^{kl}\eta_i^k\eta_j^l} \leq \overline{a}_{\mathcal{D}} < \infty,\ \xi \in \mathcal{D},\ \overline{a}_{\mathcal{D}} \geq 1, \tag{47}$$

$$\text{with } \eta_i^k = \eta_k^i \text{ for the elasticity case,}$$

$$A_{ij}(\xi) = \omega E_{ij},\ \xi \in \mathcal{D}^\perp,\ 0 < \omega \le 1.$$

For the elasticity case conditions (47) with small ω correspond to inclusions $\mathcal{D}^\perp$ of a soft material. If the material is anisotropic, then anisotropic axes must have the same directions in all inclusions $\mathcal{D}_q^\perp$.

We note, that for $\omega > 0$ conditions (47) lead (8), and all the results of Subsection 1.5 still hold here. But $\underline{a}$ tends to zero with ω, therefore there is no convergence of the iterative method (9) uniformly of $\omega \to 0$. The uniform of $\omega \to 0$ correctness estimate for the problem (46) with arbitrary $\mathbf{f}_i$ also is absent. Evidently, a norm of the solution of (46) can tend to infinity as $\omega \to 0$.

The main goal of this Section is to prove, however, a uniform of $\omega \to 0$ convergence estimate of the iterative method (9) with a special initial guess and a correctness estimate for the problem with some restrictions of $\mathbf{f}_i$.

3.2

An analog of Theorem 2.1 for $\omega > 0$ is the following

Theorem 3.1 *Let*

$$\mathbf{g}_i = \begin{cases} \text{an arbitrary function of } \mathbf{L}_2(\mathcal{D}) \text{ in } \mathcal{D}, \\ \mathbf{f}_i/\omega \text{ in } \mathcal{D}^\perp. \end{cases} \tag{48}$$

For the elasticity case the functions $\mathbf{g}_i$ must satisfy in every domain $\mathcal{D}_p$ the following conditions, cf. (31),

$$tr\{(F_p - \mathbf{w}G_p)C_p\} = 0, \tag{49}$$

where F_p and G_p are averages in $\mathcal{D}_p$ of the matrices with $\mathbf{f}_i$ and $\mathbf{g}_i$ as columns, and C_p is an arbitrary $s \times s$ matrix, $C_p = -C_p^T$, such that the restriction $C_p\varepsilon\,|_{\mathcal{D}_p}$ is continuous.

We consider the problem (46) with conditions (47). Let matrices E_{ij} satisfy requirements of Subsection 1.5 and a set $\mathcal{D}$ compiles with requirements of Subsection 1.7.

Then the problem (46) has a unique solution $\mathbf{u} \in \mathbf{H}$ and

$$\Lambda_E(\mathbf{u},\mathbf{u}) \le const\ \{\langle(\mathbf{f}_i,\mathbf{f}_i)\rangle + \langle(\mathbf{g}_i,\mathbf{g}_i)\rangle\}, \tag{50}$$

where the constant const does not depend on ω.

We now explain the condition (49) for a domain $\mathcal{D}_p$ not surrounding the torus $\mathbf{T}$ as an example. For that domain the condition (49) is equivalent, see Subsection 2.4, to

$$F_p - \omega G_p = F_p^T - \omega G_p^T. \tag{51}$$

If $F_p = F_p^T$, then one can take $\mathbf{g}_i \equiv 0$ in $\mathcal{D}$ to satisfy (51). If $F_p \ne F_p^T$, then the choice $\mathbf{g}_i = \mathbf{f}_i/\omega$ in $\mathcal{D}$ is possible to satisfy (51). But, for this choice, $\langle(\mathbf{g}_i,\mathbf{g}_i)\rangle$ of (50) tends to infinity as ω tends to zero, if $\mathbf{f}_i$ do not depend on ω. This associates with the statement that for the limit case $\omega = 0$ the condition (31), i.e., $F_p = F_p^T$, is necessary, if the domain $\mathcal{D}_p$ is not surround the torus $\mathbf{T}$.

Of interest is a solution $\mathbf{u}$ dependence of $\omega \to 0$.

Theorem 3.2 *We consider the problem (46) with conditions (47). Let matrices E_{ij} satisfy requirements of Subsection 1.5 and a set $\mathcal{D}$ compiles with requirements of Subsection 1.7. Let also the functions $\mathbf{f}_i$ be independent on ω, and, for the elasticity case, the functions $\mathbf{f}_i$ satisfy in every domain $\mathcal{D}_p$ the following conditions,*

$$tr\{F_p C_p\} = 0, \tag{52}$$

where F_p is an average in $\mathcal{D}_p$ of the matrix with $\mathbf{f}_i$ as columns, and C_p is an arbitrary $s \times s$ matrix, $C_p = -C_p^T$, such that the restriction $C_p \xi \mid_{\mathcal{D}_p}$ is continuous.

Then

$$\mathbf{u} = \omega^{-1} \mathbf{u}_K + \mathbf{O}(1) \text{ in } \mathbf{H}_E, \ \omega \to 0, \tag{53}$$

where $\mathbf{u}_K \in \mathbf{Ker}$ is a (unique) solution of the problem

$$\int_{\mathcal{D}^\perp} \left(E_{ij} \frac{\partial \mathbf{u}_K}{\partial \xi_j}, \frac{\partial \mathbf{v}}{\partial \xi_i} \right) d\mathcal{D}^\perp = \int_{\mathcal{D}^\perp} \left(\mathbf{f}_i, \frac{\partial \mathbf{v}}{\partial \xi_i} \right) d\mathcal{D}^\perp, \ \mathbf{v} \in \mathbf{Ker}, \tag{54}$$

and the subspace $\mathbf{Ker} \subset \mathbf{H}$ were defined in Subsection 2.2.

In the particular case $\mathbf{f}_i \equiv 0$ in $\mathcal{D}^\perp$ we have $\mathbf{u}_K \equiv 0$ and then

$$\mathbf{u} = \overset{0}{\mathbf{u}}_I + \mathbf{O}(\omega) \text{ in } \mathbf{H}_E, \ \omega \to 0, \tag{55}$$

where $\overset{0}{\mathbf{u}}_I \in \mathbf{Im}$ is the normal in $\mathbf{H}_E$ solution of the problem

$$\int_{\mathcal{D}} \left(A_{ij} \frac{\partial \overset{0}{\mathbf{u}}_I}{\partial \xi_j}, \frac{\partial \mathbf{v}}{\partial \xi_i} \right) d\mathcal{D} = \int_{\mathcal{D}} \left(\mathbf{f}_i, \frac{\partial \mathbf{v}}{\partial \xi_i} \right) d\mathcal{D}, \ \mathbf{v} \in \mathbf{H}, \tag{56}$$

i.e., the problem (46) with $\omega = 0$ in (47), see previous Section.

We note, that the same representation of the solution for small ω was found for Dirichlet boundary value problem of the diffusion equation.[9]

Remark 3.1 *Using $\{\overset{0}{W}{}_2^1(\mathcal{D})\}^m \subseteq \mathbf{Ker}$, we get that the solution $\mathbf{u}_K$ of the problem (54) is in $\mathcal{D}^\perp$ also a (generalized) solution of the next boundary value problem*

$$\frac{\partial}{\partial \xi_i} \left(E_{ij} \frac{\partial \mathbf{u}_K}{\partial \xi_j} - \mathbf{f}_i \right) = 0, \ \xi \in \mathcal{D}^\perp \tag{57}$$

with the following boundary conditions on every connected component Γ of the boundary $\partial \mathcal{D}^\perp$

$$\mathbf{u}_K \mid_\Gamma = \mathbf{c}_\Gamma \text{ or, for the elasticity case, } = \mathbf{c}_\Gamma + C_\Gamma \xi \mid_\Gamma, \tag{58}$$

where the vector $\mathbf{c}_\Gamma$ and the $s \times s$ matrix $C_\Gamma = -C_\Gamma^T$ both independent of ξ. Vectors $\mathbf{c}_\Gamma$ and matrices C_Γ cannot, in general, be chosen arbitrary. They are defined by the condition $\mathbf{u}_K \in \mathbf{Ker}$, i.e., $\mathbf{u}_K$ in $\mathcal{D}_p$ equals to $\mathbf{c}_p$, or, for the elasticity case, to $\mathbf{c}_p + C_p \xi$ with a constant vector $\mathbf{c}_p$ and a constant $s \times s$ matrix $C_p = -C_p^T$ such that the restriction $C_p \xi \mid_{\mathcal{D}_p}$ is continuous.

If the set $\mathcal{D}$ is connected, then $p=1$ and vectors $\mathbf{c}_\Gamma$ and matrices C_Γ must be the same for all different components Γ of the boundary $\partial\mathcal{D}=\partial\mathcal{D}^\perp$. In that case a difference between the solution $\mathbf{u}_K$ of the problem (54) and a solution of the problem (57) with homogeneous Dirichlet boundary conditions on $\partial\mathcal{D}^\perp$ equals simple to a constant, or a constant plus a rotation for the elasticity case.

In other words, we can consider the problem (46) with (47) for small ω as a problem of Fictitious Domain Method applied to homogeneous Dirichlet boundary value problem for the equation (57) in a simple connected domain $\mathcal{D}^\perp$, or in a finite collection of a simple connected domains $\mathcal{D}_q^\perp$. We can find a solution of the problem (46) with(47) on torus $\mathbf{T}$, subtract an appropriate constant vector, or a shift plus a rotation for the elasticity case, and get an $\mathbf{O}(\omega)$ in $\mathbf{H}_E$ approximation to a solution of the problem (57) with homogeneous Dirichlet boundary conditions. We even can take the limit case $\omega=0$ to obtain an exact solution of the problem (57) with homogeneous Dirichlet boundary conditions.

In the next theorem we state, that the iterative method (9) can be applied for effective solution of the problem (46) with (47) as well as of the problem (4) with (8), but the special initial guess must be chosen by analogy with previous Section.

Theorem 3.3 *Let conditions of Theorem 3.1 be satisfied. We consider the iterative method (9):*

$$\Lambda_E\left(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\tau},\mathbf{v}\right)+\Lambda_A(\mathbf{u}^n,\mathbf{v})=\left\langle\left(\mathbf{f}_i,\frac{\partial\mathbf{v}}{\partial\xi_i}\right)\right\rangle,\ \mathbf{v}\in\mathbf{H},\ n=0,1,\ldots \tag{59}$$

with the initial guess $\mathbf{u}^0$, a solution of

$$\Lambda_E(\mathbf{u}^0,\mathbf{v})=\langle(\mathbf{g}_i,\partial\mathbf{v}/\partial\xi_i)\rangle,\ \mathbf{v}\in\mathbf{H}. \tag{60}$$

For an appropriate $\tau>0$ iteration approximations $\mathbf{u}^n$ converge to a solution of the problem (46) in $\mathbf{H}_E$ with the rate of a geometric progression whose convergence factor can be bounded above by a quantity depending only on $\kappa\underline{a}_{\mathcal{D}}/\overline{a}_{\mathcal{D}}$, where $\kappa>0$ is the constant of the Proposition 2.1 of extension in $\mathbf{H}_E$ from $\mathcal{D}$ to $\mathbf{T}$.

In particular,

$$\Lambda_E(\varepsilon^n,\varepsilon^n)\le q^{2n}\Lambda_E(\varepsilon^0,\varepsilon^0),\ q=1-\kappa\underline{a}_{\mathcal{D}}/\overline{a}_{\mathcal{D}},\ \text{if}\ \tau=1/\overline{a}_{\mathcal{D}}. \tag{61}$$

For the initial error $\mathbf{u}^0-\mathbf{u}$ we have

$$\Lambda_E(\mathbf{u}^0-\mathbf{u},\mathbf{u}^0-\mathbf{u})\le const\ \{\langle(\mathbf{f}_i,\mathbf{f}_i)\rangle+\langle(\mathbf{g}_i,\mathbf{g}_i)\rangle\}, \tag{62}$$

where the constant const does not depend on ω.

Remark 3.2 *Let $\mathcal{D}$ be connected and for the elasticity case let also none of the connected components $\mathcal{D}_q^\perp$ of the complement $\mathcal{D}^\perp$ surrounds the torus $\mathbf{T}$. Then $\mathbf{Ker}=\{\overset{0}{W}{}^{1}_{2}(\mathcal{D})\}^m$ and statements of the Theorem 3.1 and Theorem 3.3 still hold, if even arbitrary constant vectors $\mathbf{c}_q$ were added to the functions $\mathbf{g}_i$ of (48) in $\mathcal{D}_q^\perp$, e.g., vectors*

$$\mathbf{c}_q=-\frac{\int_{\mathcal{D}_q^\perp}\mathbf{f}_i\,d\mathcal{D}_q^\perp}{\omega\ mes\ \mathcal{D}_q^\perp}.$$

If the right hand sides $\mathbf{f}_i$ *are not depended on* ξ *in* $\mathcal{D}^\perp$*, then using (48) and adding these constant vectors to* $\mathbf{g}_i$ *of (48) we obtain, that* $\mathbf{g}_i \equiv 0$ *are possible as well. Hence, the estimate (50) becomes an ordinary inequality of correctness, and (60) leads to the trivial initial guess* $\mathbf{u}^0 \equiv 0$.

We now consider the special case of constant coefficients $A_{ij}(\xi)$ in $\mathcal{D}$, like in previous Section.

Theorem 3.4 *Let conditions of Theorem 3.3 be satisfied. We consider the following particular case of the condition (47)*

$$A_{ij}(\xi) = E_{ij},\ \xi \in \mathcal{D}, \qquad (63)$$
$$A_{ij}(\xi) = \omega E_{ij},\ \xi \in \mathcal{D}^\perp,\ 0 < \omega \le 1,$$

and the following particular choice in (48)

$$\mathbf{g}_i = \begin{cases} \mathbf{f}_i \text{ in } \mathcal{D}, \\ \mathbf{f}_i/\omega \text{ in } \mathcal{D}^\perp \end{cases} \qquad (64)$$

to find an initial guess $\mathbf{u}^0$.

Then

$$supp\ \mathbf{r}^n = supp \frac{\partial}{\partial \xi_i}\left(E_{ij}\frac{\partial(\mathbf{u}^{n+1} - \mathbf{u}^n)}{\partial \xi_j}\right) \subseteq \partial\mathcal{D}, \qquad (65)$$

where

$$\mathbf{r}^n = \frac{\partial}{\partial \xi_i}\left(A_{ij}\frac{\partial \mathbf{u}^n}{\partial \xi_j} - \mathbf{f}_i\right) \in \{W_2^{-1}(\mathbf{T})\}^m$$

are residuals.

Proofs of these theorems base on the next important statements.

Lemma 3.1 *We consider the problem (46) with conditions (47). Let matrices* E_{ij} *satisfy requirements of Subsection 1.5 and a set* $\mathcal{D}$ *compiles with requirements of Subsection 1.7.*

Then the following decomposed representation exists

$$\Lambda_A(\mathbf{w},\mathbf{v}) = \Lambda_A(\mathbf{w}_I,\mathbf{v}_I) + \omega\Lambda_E(\mathbf{w}_K,\mathbf{v}_K),$$
$$\mathbf{w} = \mathbf{w}_I + \mathbf{w}_K \in \mathbf{H},\ \mathbf{v} = \mathbf{v}_I + \mathbf{v}_K \in \mathbf{H}, \qquad (66)$$
$$\mathbf{w}_I,\ \mathbf{v}_I \in \mathbf{Im},\ \mathbf{w}_K,\ \mathbf{v}_K \in \mathbf{Ker}.$$

Lemma 3.2 *Let conditions of Theorem 3.3 be satisfied. Then*

1. *The difference* $\mathbf{u}^0 - \mathbf{u} \in \mathbf{Im}$ *of the initial guess of (60) and the solution of the problem (46).*

2. *In the subspace* $\mathbf{Im}$ *we have*

$$0 < \kappa \underline{a}_{\mathcal{D}} \le \frac{\Lambda_A(\mathbf{v},\mathbf{v})}{\Lambda_E(\mathbf{v},\mathbf{v})} \le \bar{a}_{\mathcal{D}} < \infty,\ \mathbf{v} \in \mathbf{Im}. \qquad (67)$$

3. *Subspace* $\mathbf{Im}$ *is an invariant subspace of the operator* $L : \mathbf{H} \to \mathbf{H}$ *defined by the rule, cf. (26),*

$$\Lambda_E(L\mathbf{w},\mathbf{v}) = \Lambda_A(\mathbf{w},\mathbf{v}),\ \mathbf{w},\ \mathbf{v} \in \mathbf{H}. \qquad (68)$$

3.3

This last subsection contains proofs.

Proof of Lemma 3.1. We set

$$\Lambda_{A^0}(\mathbf{w},\mathbf{v}) = \int\limits_{\mathcal{D}} \left(A_{ij}\frac{\partial \mathbf{w}}{\partial \xi_j}, \frac{\partial \mathbf{v}}{\partial \xi_i} \right) d\mathcal{D}$$

and note that the bilinear form $\Lambda_{A^0}(\star,\star)$ equals to $\Lambda_A(\star,\star)$ with $\omega = 0$ and has a kernel **Ker**, been described in the previous Section. We have

$$\Lambda_A(\mathbf{w},\mathbf{v}) = \Lambda_{A^0}(\mathbf{w},\mathbf{v}) + \omega \int\limits_{\mathcal{D}^\perp} \left(E_{ij}\frac{\partial \mathbf{w}}{\partial \xi_j}, \frac{\partial \mathbf{v}}{\partial \xi_i} \right) d\mathcal{D}^\perp. \tag{69}$$

We insert the orthogonal in $\mathbf{H}_E$ representations

$$\mathbf{w} = \mathbf{w}_I + \mathbf{w}_K \in \mathbf{H},\ \mathbf{v} = \mathbf{v}_I + \mathbf{v}_K \in \mathbf{H},$$

$$\mathbf{w}_I,\ \mathbf{v}_I \in \mathbf{Im},\ \mathbf{w}_K,\ \mathbf{v}_K \in \mathbf{Ker}$$

into the right hand side of (69). The first term

$$\Lambda_{A^0}(\mathbf{w},\mathbf{v}) = \Lambda_{A^0}(\mathbf{w}_I,\mathbf{v}_I),$$

because **Ker** is a kernel of this bilinear form. For the second term we have to prove that

$$\int\limits_{\mathcal{D}^\perp} \left(E_{ij}\frac{\partial \mathbf{w}}{\partial \xi_j}, \frac{\partial \mathbf{v}}{\partial \xi_i} \right) d\mathcal{D}^\perp =$$

$$\int\limits_{\mathcal{D}^\perp} \left(E_{ij}\frac{\partial \mathbf{w}_I}{\partial \xi_j}, \frac{\partial \mathbf{v}_I}{\partial \xi_i} \right) d\mathcal{D}^\perp + \int\limits_{\mathcal{D}^\perp} \left(E_{ij}\frac{\partial \mathbf{w}_K}{\partial \xi_j}, \frac{\partial \mathbf{v}_K}{\partial \xi_i} \right) d\mathcal{D}^\perp.$$

The equality above follows from

$$\int\limits_{\mathcal{D}^\perp} \left(E_{ij}\frac{\partial \mathbf{w}_I}{\partial \xi_j}, \frac{\partial \mathbf{v}_K}{\partial \xi_i} \right) d\mathcal{D}^\perp = 0,\ \mathbf{w}_I \in \mathbf{Im},\ \mathbf{v}_K \in \mathbf{Ker}. \tag{70}$$

Equality (70) in its turn is checked directly,

$$\int\limits_{\mathcal{D}^\perp} \left(E_{ij}\frac{\partial \mathbf{w}_I}{\partial \xi_j}, \frac{\partial \mathbf{v}_K}{\partial \xi_i} \right) d\mathcal{D}^\perp = \Lambda_E(\mathbf{w}_I,\mathbf{v}_K) - \int\limits_{\mathcal{D}} \left(E_{ij}\frac{\partial \mathbf{w}_I}{\partial \xi_j}, \frac{\partial \mathbf{v}_K}{\partial \xi_i} \right) d\mathcal{D} = 0.$$

Here both terms equal to zero, due to $\mathbf{H}_E$ orthogonality of $\mathbf{w}_I$ and $\mathbf{v}_K$, and $\mathbf{v}_K \in \mathbf{Ker}$. Π

Proof of Lemma 3.2.

1. By definition of the subspace **Im** we have to verify the equality

$$\Lambda_E(\mathbf{u}^0 - \mathbf{u}, \mathbf{v}) = 0,\ \mathbf{v} \in \mathbf{Ker}.$$

We will use the following immediate consequence of Lemma 3.1,

$$\Lambda_A(\mathbf{w},\mathbf{v}) = \omega\Lambda_E(\mathbf{w},\mathbf{v}),\ \mathbf{w}\in\mathbf{H},\ \mathbf{v}\in\mathbf{Ker}. \tag{71}$$

We put in (71) $\mathbf{w}=\mathbf{u}$, the solution of (46), and obtain

$$\Lambda_E(\mathbf{u},\mathbf{v}) = \frac{1}{\omega}\Lambda_A(\mathbf{u},\mathbf{v}) = \left\langle \left(\frac{\mathbf{f}_i}{\omega}, \frac{\partial \mathbf{v}}{\partial \xi_i}\right)\right\rangle, \mathbf{v}\in\mathbf{Ker}.$$

Subtracting by parts this equality from (60) gives

$$\Lambda_E(\mathbf{u}^0-\mathbf{u},\mathbf{v}) = \left\langle \left(\mathbf{g}_i - \frac{\mathbf{f}_i}{\omega}, \frac{\partial \mathbf{v}}{\partial \xi_i}\right)\right\rangle, \mathbf{v}\in\mathbf{Ker}.$$

In according to (48) in right hand side $\mathbf{g}_i - \mathbf{f}_i/\omega \equiv 0$ in $\mathcal{D}^\perp$ and integrals

$$\int\limits_{\mathcal{D}_p} \left(\mathbf{g}_i - \frac{\mathbf{f}_i}{\omega}, \frac{\partial \mathbf{v}}{\partial \xi_i}\right)\, d\mathcal{D}_p,\ \mathbf{v}=\mathbf{c}_p,\ \text{or } \mathbf{c}_p + C_p\xi \text{ for the elasticity case,}$$

vanish because of condition (49), see Subsection 2.4.

2. We consider the representation (69) with $\mathbf{w}=\mathbf{v}$. Lower and upper estimates for the first term in the right hand side of (69) were obtained of Lemma 2.2. By $0<\omega\le 1\le \bar{a}_{\mathcal{D}}$, the last term is estimated trivially,

$$0\le \omega\int\limits_{\mathcal{D}^\perp}\left(E_{ij}\frac{\partial \mathbf{v}}{\partial \xi_j}, \frac{\partial \mathbf{v}}{\partial \xi_i}\right) d\mathcal{D}^\perp \le \bar{a}_{\mathcal{D}}\int\limits_{\mathcal{D}^\perp}\left(E_{ij}\frac{\partial \mathbf{v}}{\partial \xi_j}, \frac{\partial \mathbf{v}}{\partial \xi_i}\right) d\mathcal{D}^\perp.$$

3. By the definition (68) of the operator L we have to check the last equality of

$$\Lambda_E(L\mathbf{w},\mathbf{v}) = \Lambda_A(\mathbf{w},\mathbf{v}) = 0,\ \mathbf{w}\in\mathbf{Im},\ \mathbf{v}\in\mathbf{Ker}. \tag{72}$$

But this is just a particular case of the equality (71) for $\mathbf{w}\in\mathbf{Im}$.

Π

We note, like in previous Section, that the operator $L:\mathbf{H}\to\mathbf{H}$ is selfadjoint in $\mathbf{H}_E$ and inequalities (67) are equivalent to the following operator inequalities in $\mathbf{H}_E$

$$\kappa\underline{a}_{\mathcal{D}} I \le L \le \bar{a}_{\mathcal{D}} I \text{ in } \mathbf{Im}\subset\mathbf{H}_E. \tag{73}$$

Proof of Theorem 3.1. We introduce function $\mathbf{u}^0$ as a solution of the problem (60) with $\mathbf{g}_i$ of (48) and consider the difference $\mathbf{u}^0-\mathbf{u}$ of this function and the solution $\mathbf{u}$ of the original problem (46). We write

$$\Lambda_A(\mathbf{u}^0-\mathbf{u},\mathbf{v}) = \Lambda_A(\mathbf{u}^0,\mathbf{v}) - \left\langle\left(\mathbf{f}_i, \frac{\partial \mathbf{v}}{\partial \xi_i}\right)\right\rangle, \mathbf{v}\in\mathbf{H} \tag{74}$$

and present vector $\mathbf{v}\in\mathbf{H}$ here as an orthogonal in $\mathbf{H}_E$ sum,

$$\mathbf{v} = \mathbf{v}_K + \mathbf{v}_I,\ \mathbf{v}_K\in\mathbf{Ker},\ \mathbf{v}_I\in\mathbf{Im}.$$

We shell now see that all terms with $\mathbf{v}_K$ of (74) vanish.

For the left hand side, $\Lambda_A(\mathbf{u}^0-\mathbf{u},\mathbf{v})=0$ because of (72), as $\mathbf{u}^0-\mathbf{u}\in\mathbf{Im}$ by Lemma 3.2 and $\mathbf{v}_K\in\mathbf{Ker}$. For the right hand side our checking becomes more complicated.

In according to (71) we rewrite the first term with $\mathbf{v}=\mathbf{v}_K$,

$$\Lambda_A(\mathbf{u}^0,\mathbf{v}_K)=\omega\Lambda_E(\mathbf{u}^0,\mathbf{v}_K)=\omega\left\langle\left(\mathbf{g}_i,\frac{\partial\mathbf{v}}{\partial\xi_i}\right)\right\rangle,\mathbf{v}_K\in\mathbf{Ker},$$

using the definition (60) of $\mathbf{u}^0$ for the last equality. Making that substitution we obtain the right hand side of (74) in the form $\langle(\omega\mathbf{g}_i-\mathbf{f}_i,\partial\mathbf{v}_K/\partial\xi_i)\rangle$, and this value is zero owing to the condition (49).

Thus, we have just prove that is sufficient to put $\mathbf{v}\in\mathbf{Im}$ in (74). Lemma 3.2 states that $\mathbf{u}^0-\mathbf{u}\in\mathbf{Im}$ and the symmetric bilinear form $\Lambda_A(\star,\star)$ is $\mathbf{H}_E$ bounded and $\mathbf{H}_E$ coercivity in the subspace $\mathbf{Im}$ uniformly of $\omega>0$.

We now estimate uniformly of ω the $\mathbf{H}_E$ norm of the linear functional of the right hand side of (74). We will not even use $\mathbf{v}\in\mathbf{Im}$ here.

For the first term,

$$|\Lambda_A(\mathbf{u}^0,\mathbf{v})|^2\le\Lambda_A(\mathbf{u}^0,\mathbf{u}^0)\Lambda_A(\mathbf{v},\mathbf{v})\le\vec{a}_{\mathcal{D}}^2\Lambda_E(\mathbf{u}^0,\mathbf{u}^0)\Lambda_E(\mathbf{v},\mathbf{v}),$$

where

$$\Lambda_E(\mathbf{u}^0,\mathbf{u}^0)\le\langle(\mathbf{g}_i,\mathbf{g}_i)\rangle \tag{75}$$

with a constant *const* independent of ω.

For the second one,

$$|\langle(\mathbf{f}_i,\partial\mathbf{v}/\partial\xi_i)\rangle|^2\le const\langle(\mathbf{f}_i,\mathbf{f}_i)\rangle\Lambda_E(\mathbf{v},\mathbf{v}),\ \mathbf{v}\in\mathbf{H},$$

with a constant *const* independent of ω as well.

We now combine these estimates,

$$\left|\Lambda_A(\mathbf{u}^0,\mathbf{v})-\left\langle\left(\mathbf{f}_i,\frac{\partial\mathbf{v}}{\partial\xi_i}\right)\right\rangle\right|^2\le const\{\langle(\mathbf{f}_i,\mathbf{f}_i)\rangle+\langle(\mathbf{g}_i,\mathbf{g}_i)\rangle\}\Lambda_E(\mathbf{v},\mathbf{v}).$$

Therefore, the problem (74) is well posed in the subspace $\mathbf{Im}$ of $\mathbf{H}_E$ uniformly of ω,

$$\Lambda_E(\mathbf{u}^0-\mathbf{u},\mathbf{u}^0-\mathbf{u})\le const\{\langle(\mathbf{f}_i,\mathbf{f}_i)\rangle+\langle(\mathbf{g}_i,\mathbf{g}_i)\rangle\} \tag{76}$$

with a constant *const* independent of ω.

By triangle inequality we conclude from (76) using (75), that the desired estimate (50) holds. Π

Proof of Theorem 3.2. There is a special representation (66) of the bilinear form $\Lambda_A(\star,\star)$. We write the same representation for the linear functional of the right hand side,

$$\left\langle\left(\mathbf{f}_i,\frac{\partial\mathbf{v}}{\partial\xi_i}\right)\right\rangle=\left\langle\left(\mathbf{f}_i,\frac{\partial\mathbf{v}_I}{\partial\xi_i}\right)\right\rangle+\left\langle\left(\mathbf{f}_i,\frac{\partial\mathbf{v}_K}{\partial\xi_i}\right)\right\rangle, \tag{77}$$

$$\mathbf{v}=\mathbf{Ker}_I+\mathbf{v}_I,\ \mathbf{v}_I\in\mathbf{Im},\ \mathbf{v}_K\in\mathbf{Ker}.$$

Then the original problem falls into two following independent problems

$$\Lambda_A(\mathbf{u}_I,\mathbf{v}_I) = \left\langle \left(\mathbf{f}_i, \frac{\partial \mathbf{v}_I}{\partial \xi_i} \right) \right\rangle, \mathbf{u}_I,\ \mathbf{v}_I \in \mathbf{Im}, \tag{78}$$

$$\Lambda_A(\mathbf{u}_K,\mathbf{v}_K) = \left\langle \left(\mathbf{f}_i, \frac{\partial \mathbf{v}_K}{\partial \xi_i} \right) \right\rangle, \mathbf{u}_K,\ \mathbf{v}_K \in \mathbf{Ker}, \tag{79}$$

and the solution $\mathbf{u}$ of (46) is an $\mathbf{H}_E$ orthogonal sum,

$$\mathbf{u} = \mathbf{u}_I + \frac{\mathbf{u}_K}{\omega}, \tag{80}$$

where $\mathbf{u}_I$ and $\mathbf{u}_K$ are solutions of (78) and (79).

We first consider (53). We will prove that the problem (79) is equivalent to (54) with conditions (52), has a unique solution, and that the term $\mathbf{u}_I$ of (80) is bounded in $\mathbf{H}_E$ uniformly of ω. These statements lead to (53).

The equivalence of (79) and (54) is verified directly. The condition (52) is involved for the elasticity case in the last of the following equalities,

$$\int_{\mathcal{D}} \left(\mathbf{f}_i, \frac{\partial \mathbf{v}}{\partial \xi_i} \right) d\mathcal{D} = \sum_p \int_{\mathcal{D}_p} \left(\mathbf{f}_i, \frac{\partial \mathbf{v}}{\partial \xi_i} \right) d\mathcal{D}_p = 0,\ \mathbf{v} = \mathbf{c}_p + C_p \xi$$

where a vector $\mathbf{c}_p$ and an $s \times s$ matrix $C_p = -C_p^T$, both independent of ξ, and restriction $C\xi\,|_{\mathcal{D}}$ is continuous in each component of connectedness $\mathcal{D}_p$, see Subsection 2.4.

The problem (79) is evidently well posed, the unique solution exists and equal to the $\mathbf{H}_E$ orthoprojection to the subspace $\mathbf{Ker}$ of the canonical representation in $\mathbf{H}_E$ of the linear bounded functional $\left\langle \left(\mathbf{f}_i, \frac{\partial \mathbf{v}_K}{\partial \xi_i} \right) \right\rangle$, $\mathbf{v}_K \in \mathbf{H}_E$.

To obtain a uniform of ω estimate for $\mathbf{H}_E$ norm of the function $\mathbf{u}_I$ we consider the problem (78). As we have noted in the Proof of Theorem 3.1, by Lemma 3.2 the symmetric bilinear form $\Lambda_A(\star,\star)$ is $\mathbf{H}_E$ bounded and $\mathbf{H}_E$ coercivity in the subspace $\mathbf{Im}$ with constants $\bar{a}_{\mathcal{D}}$ and $\kappa \underline{a}_{\mathcal{D}}$, i.e., uniformly of $\omega > 0$. Therefore

$$\Lambda_E(\mathbf{u}_I,\mathbf{u}_I) \leq const\, \langle (\mathbf{f}_i, \mathbf{f}_i) \rangle, \tag{81}$$

where *const* and $\mathbf{f}_i$ are independent of ω, and (53) is completely proved.

We now consider the particular case $\mathbf{f}_i \equiv 0$ in $\mathcal{D}^\perp$, when $\mathbf{u}_K \equiv 0$, and will verify that the term in the series of powers of ω for $\mathbf{u}_I$, that corresponds to the zero power, is the function $\overset{0}{\mathbf{u}}_I$ of (56). This statement constitutes the second part of the theorem.

It was discovered in Theorem 2.1 that the problem (56) under the conditions (52) is well posed in the subspace $\mathbf{Im} \subset \mathbf{H}_E$, has a unique by Lemma 2.1 solution $\overset{0}{\mathbf{u}}_I \in \mathbf{Im}$, and is possible to choose $\mathbf{v} \in \mathbf{Im}$ in (56) instead of $\mathbf{v} \in \mathbf{H}$. With $\mathbf{f}_i \equiv 0$ in $\mathcal{D}^\perp$ we obtain from (56) and (78) the following equation for the difference $\overset{0}{\mathbf{u}}_I - \mathbf{u}_I \in \mathbf{Im}$:

$$\Lambda_A(\overset{0}{\mathbf{u}}_I - \mathbf{u}_I, \mathbf{v}) = \omega \int_{\mathcal{D}^\perp} \left(E_{ij} \frac{\partial \overset{0}{\mathbf{u}}_I}{\partial \xi_j}, \frac{\partial \mathbf{v}}{\partial \xi_i} \right) d\mathcal{D}^\perp,\ \mathbf{v} \in \mathbf{Im}. \tag{82}$$

In the right hand side of (82) there is a linear functional bounded in $\mathbf{H}_E$ by $O(\omega)$. However, we have mentioned several times already, that an equation with the bilinear form $\Lambda_A(\star,\star)$ is well posed in the subspace $\mathbf{Im} \subset \mathbf{H}_E$ uniformly of ω. Hence, $\overset{0}{\mathbf{u}}_I - \mathbf{u}_I = \mathbf{O}(\omega)$, cf. (55). Π

Proof of Theorem 3.3. It coincides very closely with Proof of Theorem 2.2. The estimate (62) was already arrived as (76) of Proof of Theorem 3.1. Π

Proof of Theorem 3.4. We first note, that

$$\tau \mathbf{r}^n = \frac{\partial}{\partial \xi_i}\left(E_{ij}\frac{\partial(\mathbf{u}^{n+1}-\mathbf{u}^n)}{\partial \xi_j}\right)$$

exactly like in Proof of Theorem 2.3. Also, it was found there, using the inclusion $\{\overset{0}{W}{}^1_2(\mathcal{D}^\perp)\}^m \subseteq \mathbf{Ker}$, that

$$supp\frac{\partial}{\partial \xi_i}\left(E_{ij}\frac{\partial \mathbf{v}}{\partial \xi_j}\right) \subseteq \overline{\mathcal{D}},\ \mathbf{v} \in \mathbf{Im}.$$

In according to Lemma 3.2 $\mathbf{u}^0 - \mathbf{u} \in \mathbf{Im}$, and from $\mathbf{u}^n - \mathbf{u} \in \mathbf{Im}$ it follows $\mathbf{u}^{n+1} - \mathbf{u} \in \mathbf{Im}$. Therefore, $\mathbf{u}^{n+1} - \mathbf{u}^n \in \mathbf{Im}$, and then the difference $\mathbf{u}^{n+1} - \mathbf{u}^n$ can be put in the inclusion above instead of $\mathbf{w} \in \mathbf{Im}$, i.e.,

$$supp\frac{\partial}{\partial \xi_i}\left(E_{ij}\frac{\partial \mathbf{u}^{n+1}-\mathbf{u}^n}{\partial \xi_j}\right) \subseteq \overline{\mathcal{D}}. \tag{83}$$

We have now to verify

$$supp\frac{\partial}{\partial \xi_i}\left(E_{ij}\frac{\partial \mathbf{u}^{n+1}-\mathbf{u}^n}{\partial \xi_j}\right) \subseteq \overline{\mathcal{D}^\perp} = \mathbf{T}\setminus\mathcal{D}. \tag{84}$$

Let $\mathbf{Ker}^\perp \subset \mathbf{H}$ be a subspace, consists of vector functions equals to $\mathbf{c}_q$, or, for the elasticity case, to $\mathbf{c}_q + C_q\xi$ in every $\mathcal{D}_q^\perp$, where a vector $\mathbf{c}_q$ and an $s \times s$ matrix $C_q = -C_q^T$, are both independent of $\xi \in \mathcal{D}_q^\perp$ and restriction $C_q\xi\mid_{\mathcal{D}}$ is continuous in each component of connectedness $\mathcal{D}_q^\perp$ of $\mathcal{D}^\perp$.

This definition differs from the definition of $\mathbf{Ker}$ no more then the domains $\mathcal{D}_q^\perp$ play the role of the domains $\mathcal{D}_p$.

Let $\mathbf{Im}^\perp$ be an orthogonal complement of $\mathbf{Ker}^\perp$ in $\mathbf{H}_E$.

Following the Proof of Theorem 2.3 and using $\{\overset{0}{W}{}^1_2(\mathcal{D})\}^m \subseteq \mathbf{Ker}^\perp$ we find, that

$$supp\frac{\partial}{\partial \xi_i}\left(E_{ij}\frac{\partial \mathbf{v}}{\partial \xi_j}\right) \subseteq \overline{\mathcal{D}^\perp} = \mathbf{T}\setminus\mathcal{D},\ \mathbf{v} \in \mathbf{Im}^\perp.$$

By conditions of the theorem, $A_{ij}(\xi) = E_{ij}$ and $\mathbf{g}_i = \mathbf{f}_i$ in an every domain $\mathcal{D}_p$. By analogy with arguments of Lemma 3.2 we can state, that $\mathbf{u}^0 - \mathbf{u} \in \mathbf{Im}^\perp$, and from $\mathbf{u}^n - \mathbf{u} \in \mathbf{Im}^\perp$ it follows $\mathbf{u}^{n+1} - \mathbf{u} \in \mathbf{Im}^\perp$. Therefore, $\mathbf{u}^{n+1} - \mathbf{u}^n \in \mathbf{Im}^\perp$, and (84) holds.

We note, that in this proof of (84) the condition $A_{ij}(\xi) = \omega E_{ij}$ in $\mathcal{D}^\perp$ with $\omega > 0$ was not involved. We conclude, that (84) is true for $\omega = 0$ as well, cf. (45). Π

4 Composites with inclusions of soft materials and with cavities

4.1

We consider a problem (4):

$$\Lambda_A(\mathbf{u},\mathbf{v}) = \langle(\mathbf{f}_i, \partial\mathbf{v}/\partial\xi_i)\rangle,\ \mathbf{v}\in\mathbf{H}, \tag{85}$$

in conditions (11), (13), i.e.

$$0 < \underline{a}_{\mathcal{D}} \le \frac{\sum a_{ij}^{kl}(\xi)\eta_i^k\eta_j^l}{\sum e_{ij}^{kl}\eta_i^k\eta_j^l} \le \bar{a}_{\mathcal{D}} < \infty,\ \xi\in\mathcal{D},\ \bar{a}_{\mathcal{D}} \ge 1, \tag{86}$$

$$\text{with } \eta_i^k = \eta_k^i \text{ for the elasticity case,}$$

$$A_{ij}(\xi) = \omega_q E_{ij},\ \xi\in\mathcal{D}_q^\perp,\ 0\le\omega_q\le 1.$$

For the elasticity case conditions (86) with small ω_q correspond to inclusions $\mathcal{D}^\perp$ of soft materials. The materials must be of the same type, i.e., their tensors of elastic modulus may differ by constant multiplicands ω_q. If the inclusion materials are anisotropic, then anisotropic axes must have the same directions in all inclusions $\mathcal{D}_q^\perp$. Possibility of $\omega_q = 0$ corresponds to a cavity $\mathcal{D}_q^\perp$, like in Section 2.

We have studied the case all $\omega_q = \omega$ in the previous Section. Here we consider peculiarities of the multi-parameter problem (86). There will be specially mentioned situations, when we suppose for simplicity, that

$$\omega_0 = 0,\ \omega_{q_1} \ne \omega_{q_2},\ q_1 \ne q_2. \tag{87}$$

4.2

In this Subsection we define "Condition of representation of constants or, for the elasticity case, shifts and rotations in $\mathcal{D}\subset\mathbf{T}$," which will play an important role for proving the statements of this Section. We constructed an example showing, that the condition is necessary for Theorem 4.1 and for the estimate (96) of Theorem 4.2. We expect, however, that Theorem 4.2, except (96), and Theorem 4.3 can be proved without that condition, as well as Lemma 4.1. But it is beyond the scope of our present paper.

Suppose (87) holds.

For every $\mathcal{D}_q^\perp\subset\mathbf{T}$ we denote $\mathbf{Ker}_q\subset\mathbf{H}$, a subspace of vector functions independent of ξ, or, for the elasticity case, equal to a shift plus a rotation in every connected component of the set $\mathbf{T}\setminus\overline{\mathcal{D}_q^\perp}$. If $\mathcal{D}^\perp$ is connected, then $\mathcal{D}^\perp = \mathcal{D}_q^\perp$ and $\mathbf{Ker} = \mathbf{Ker}_q$, see Section 2 for the definition of $\mathbf{Ker}$.

In this Section we consider the case when $\mathcal{D}^\perp$ is not connected. Therefore, there are different subspaces $\mathbf{Ker}_q$ for different connected components $\mathcal{D}_q^\perp$ of $\mathcal{D}^\perp$.

Definition 4.1 *Condition of representation of constants or, for the elasticity case, shifts and rotations in $\mathcal{D}\subset\mathbf{T}$ is the following*

$$\mathbf{Ker} = \sum_q \mathbf{Ker}_q. \tag{88}$$

If (87) is not hold, i.e., several of ω_q are the same, then the corresponded domains $\mathcal{D}_q^{\perp}$ must be united into a single set $\mathcal{D}_q^{\perp}$ and, further, definition of $\mathbf{Ker}_q$ and condition (88) are still the same. For example, if all ω_q are the same, we unite all $\mathcal{D}_q^{\perp}$ into one $\mathcal{D}_q^{\perp}$. However, then $\mathcal{D}_q^{\perp} = \mathcal{D}^{\perp}$ and (88) holds trivially.

Another simple possibility to satisfy the condition (88) is a connected domain $\mathcal{D}$ except the elasticity case. For the elasticity case both the domain $\mathcal{D}$ and its image of the canonical expansion of $\mathbf{T}$ into $\mathbf{R}^s$ must be connected to prove (88) simply.

In a general (except the elasticity) case it is only easy to proof

$$\mathbf{Ker} \supseteq \sum_q \mathbf{Ker}_q, \tag{89}$$

but for the elasticity case we did not check (89).

A simple example, that condition (88) is not always true, has been kindly presented by S.P. Novikov. Let $s = 2,\ m = 1$, and

$$\mathcal{D} = \left\{ \xi \in \mathbf{T} : 0 < \xi_1 < \frac{1}{4} \cup \frac{1}{2} < \xi_1 < \frac{3}{4} \right\},$$

i.e., $\mathcal{D}$ consists of two connected domains, both of them surround the torus $\mathbf{T}$ along the direction of ξ_2 axis, and $\mathcal{D}^{\perp}$ has the same structure. Functions of $\mathbf{Ker}$ may achieve two different constant values in $\mathcal{D}_1$ and $\mathcal{D}_2$, however, functions of $\mathbf{Ker}_1$ and $\mathbf{Ker}_2$, therefore, and their sum can only achieve the same constant values in $\mathcal{D}_1$ and $\mathcal{D}_2$.

4.3

A multi-parameter analog of Theorem 3.1 is the following

Theorem 4.1 *Let*

$$\mathbf{g}_i = \begin{cases} \text{an arbitrary function of } \mathbf{L}_2(\mathcal{D}) \text{ in } \mathcal{D}, \\ \mathbf{0} \text{ in } \mathcal{D}_q^{\perp}, \text{ where } \omega_q = 0, \\ \mathbf{f}_i/\omega_q \text{ in } \mathcal{D}_q^{\perp}, \text{ where } \omega_q > 0. \end{cases} \tag{90}$$

For the elasticity case the functions $\mathbf{f}_i$ and $\mathbf{g}_i$ must satisfy in every domain $\mathcal{D}_p$ and $\mathcal{D}_q^{\perp}$ the following conditions, cf. (31),

$$trFC = 0,\ trGC = 0, \tag{91}$$

where $s \times s$ matrices F and G are averages in a given domain $\mathcal{D}_p$ or $\mathcal{D}_q^{\perp}$ of the matrices with $\mathbf{f}_i$ and $\mathbf{g}_i$ as columns, and C is an arbitrary $s \times s$ matrix, $C_p = -C_p^T$, such that the restriction $C\varepsilon$ on a given domain $\mathcal{D}_p$ or $\mathcal{D}_q^{\perp}$ is continuous.

We consider the problem (85) with conditions (86). We suppose (88) is true. Let matrices E_{ij} satisfy requirements of Subsection 1.5 and a set $\mathcal{D}$ compiles with requirements of Subsection 1.7.

Let also all $\mathbf{f}_i \equiv 0$ in the domains $\mathcal{D}_q^{\perp}$ with $\omega_q = 0$.

Then the problem (85) has a unique normal in $\mathbf{H}_E$ solution $\mathbf{u} \in \mathbf{H}_E$ and

$$\Lambda_E(\mathbf{u},\mathbf{u}) \le const\ \{\langle(\mathbf{f}_i,\mathbf{f}_i)\rangle + \langle(\mathbf{g}_i,\mathbf{g}_i)\rangle\}, \tag{92}$$

with the constant const independent of the collection $\{\omega_q\}$.

We did not try to generalize Theorem 3.2 on a solution dependence of $\omega \to 0$, but we are sure that is possible.

All other statements are extended with few modifications.

Theorem 4.2 *Let conditions of Theorem 4.1 be satisfied. We consider the iterative method (9):*

$$\Lambda_E\left(\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\tau},\mathbf{v}\right)+\Lambda_A(\mathbf{u}^n,\mathbf{v})=\left\langle\left(\mathbf{f}_i,\frac{\partial\mathbf{v}}{\partial\xi_i}\right)\right\rangle,\ \mathbf{v}\in\mathbf{H},\ n=0,1,\ldots \tag{93}$$

with the initial guess $\mathbf{u}^0$, *a solution of*

$$\Lambda_E(\mathbf{u}^0,\mathbf{v})=\langle(\mathbf{g}_i,\partial\mathbf{v}/\partial\xi_i)\rangle,\ \mathbf{v}\in\mathbf{H}. \tag{94}$$

For an appropriate $\tau>0$ *iteration approximations* $\mathbf{u}^n$ *converge to a solution of the problem (85) in* $\mathbf{H}_E$ *with the rate of a geometric progression whose convergence factor can be bounded above by a quantity depending only on* $\kappa\underline{a}_{\mathcal{D}}/\overline{a}_{\mathcal{D}}$, *where* $\kappa>0$ *is the constant of the Proposition 2.1 of extension in* $\mathbf{H}_E$ *from* $\mathcal{D}$ *to* $\mathbf{T}$.

In particular,

$$\Lambda_E(\varepsilon^n,\varepsilon^n)\le q^{2n}\Lambda_E(\varepsilon^0,\varepsilon^0),\ q=1-\kappa\underline{a}_{\mathcal{D}}/\overline{a}_{\mathcal{D}},\ \textit{if}\ \tau=1/\overline{a}_{\mathcal{D}}. \tag{95}$$

For the initial error $\mathbf{u}^0-\mathbf{u}$ *we have the estimate*

$$\Lambda_E(\mathbf{u}^0-\mathbf{u},\mathbf{u}^0-\mathbf{u})\le const\ \{\langle(\mathbf{f}_i,\mathbf{f}_i)\rangle+\langle(\mathbf{g}_i,\mathbf{g}_i)\rangle\}, \tag{96}$$

where the constant const does not depend on $\{\omega_q\}$.

Remark 4.1 *Let* $\mathcal{D}$ *be connected and for the elasticity case let also none of the connected components* $\mathcal{D}_q^\perp$ *of the complement* $\mathcal{D}^\perp$ *surrounds the torus* $\mathbf{T}$. *Then* $\mathbf{Ker}=\{\overset{0}{W}\frac{1}{2}(\mathcal{D})\}^m$ *and statements of the Theorem 4.1 and Theorem 4.2 still hold, if even arbitrary constant vectors* $\mathbf{c}_q$ *were added to the functions* $\mathbf{g}_i$ *of (90) in* $\mathcal{D}_q^\perp$, *e.g., vectors*

$$\mathbf{c}_q=-\frac{\int_{\mathcal{D}_q^\perp}\mathbf{f}_i d\mathcal{D}_q^\perp}{\omega_q\ mes\ \mathcal{D}_q^\perp}.$$

If the right hand sides $\mathbf{f}_i$ *are not depended on* ξ *in* $\mathcal{D}^\perp$, *then using (90) and adding these constant vectors to* $\mathbf{g}_i$ *of (90) we obtain, that* $\mathbf{g}_i\equiv 0$ *are possible as well. Hence, the estimate (89) becomes an ordinary well posed inequality, and (99) leads to the trivial initial guess* $\mathbf{u}^0\equiv 0$.

Theorem 4.3 *Let conditions of Theorem 4.2 be satisfied. We consider the following particular case of the condition (86)*

$$A_{ij}(\xi)=\omega_p E_{ij},\ \xi\in\mathcal{D}_p,\ \underline{a}_{\mathcal{D}}\le\omega_p\le\overline{a}_{\mathcal{D}}\ge 1,$$
$$A_{ij}(\xi)=\omega_q E_{ij},\ \xi\in\mathcal{D}_q^\perp,\ 0\le\omega_q\le 1, \tag{97}$$

with sets of constants $\{\omega_p\}$, $\{\omega_p\}$ *and the following particular choice in (87)*

$$\mathbf{g}_i = \begin{cases} \mathbf{f}_i/\omega_p \text{ in } \mathcal{D}_p, \\ \mathbf{0} \text{ in } \mathcal{D}_q^{\perp} \text{ with } \omega_q = 0, \\ \mathbf{f}_i/\omega_q \text{ in } \mathcal{D}^{\perp} \text{ with } \omega_q > 0 \end{cases} \tag{98}$$

to find an initial guess $\mathbf{u}^0$ *of (94).*

Then

$$\mathit{supp}\ \mathbf{r}^n = \mathit{supp} \frac{\partial}{\partial \xi_i} \left(E_{ij} \frac{\partial (\mathbf{u}^{n+1} - \mathbf{u}^n)}{\partial \xi_j} \right) \subseteq \partial \mathcal{D}, \tag{99}$$

where

$$\mathbf{r}^n = \frac{\partial}{\partial \xi_i} \left(A_{ij} \frac{\partial \mathbf{u}^n}{\partial \xi_j} - \mathbf{f}_i \right) \in \{W_2^{-1}(\mathbf{T})\}^m$$

are residuals.

Proofs base on the next important statements.

Lemma 4.1 *Let conditions of Theorem 4.2 be satisfied and the subspace* **Im** *be defined as in Section 2.*

Then

1. *The difference* $\mathbf{u}^0 - \mathbf{u} \in \mathbf{Im}$ *of the initial guess of (94) and the solution of the problem (85).*

2. *In the subspace* **Im** *we have*

$$0 < \kappa \underline{a}_{\mathcal{D}} \leq \frac{\Lambda_A(\mathbf{v}, \mathbf{v})}{\Lambda_E(\mathbf{v}, \mathbf{v})} \leq \bar{a}_{\mathcal{D}} < \infty, \ \mathbf{v} \in \mathbf{Im}. \tag{100}$$

3. *Subspace* **Im** *is an invariant subspace of the operator* $L : \mathbf{H} \to \mathbf{H}$ *defined by the rule, cf. (26),*

$$\Lambda_E(L\mathbf{w}, \mathbf{v}) = \Lambda_A(\mathbf{w}, \mathbf{v}), \ \mathbf{w}, \ \mathbf{v} \in \mathbf{H}. \tag{101}$$

4.4

This last subsection contains proofs as usual.

Proof of Lemma 4.1.

For every subspace $\mathbf{Ker}_q$ defined of Subsection 4.2 let $\mathbf{Im}_q$ be the $\mathbf{H}_E$ orthogonal complement to $\mathbf{Ker}_q$. By (88) $\mathbf{Ker} = \sum_q \mathbf{Ker}_q$, therefore

$$\mathbf{Im} = \bigcap_q \mathbf{Im}_q. \tag{102}$$

By analogy with (71) we have

$$\Lambda_A(\mathbf{w}, \mathbf{v}) = \omega_q \Lambda_E(\mathbf{w}, \mathbf{v}), \ \mathbf{w} \in \mathbf{H}, \ \mathbf{v} \in \mathbf{Ker}_q, \tag{103}$$

that can be checked immediately,

$$\Lambda_A(\mathbf{w},\mathbf{v}) - \omega_q \Lambda_E(\mathbf{w},\mathbf{v}) = \int\limits_{\mathbf{T}\setminus\overline{\mathcal{D}_q^{\perp}}} \left([A_{ij} - \omega_q E_{ij}] \frac{\partial \mathbf{w}}{\partial \xi_j}, \frac{\partial \mathbf{v}}{\partial \xi_i} \right) d\xi = 0.$$

1. By definition of the subspace $\mathbf{Im}_q$ we have to verify the equality

$$\Lambda_E(\mathbf{u}^0 - \mathbf{u}, \mathbf{v}) = 0, \ \mathbf{v} \in \mathbf{Ker}_q$$

to prove that $\mathbf{u}^0 - \mathbf{u} \in \mathbf{Im}_q$. The collection of that equalities for all $\mathbf{Ker}_q$ in according to (102) leads to the desired statement.

We first consider a domain $\mathcal{D}_q^{\perp}$ with $\omega_q > 0$.

We put in (103) $\mathbf{w} = \mathbf{u}$, the solution of (85), and obtain

$$\Lambda_E(\mathbf{u},\mathbf{v}) = \frac{1}{\omega_q} \Lambda_A(\mathbf{u},\mathbf{v}) = \left\langle \left(\frac{\mathbf{f}_i}{\omega_q}, \frac{\partial \mathbf{v}}{\partial \xi_i} \right) \right\rangle, \ \mathbf{v} \in \mathbf{Ker}_q.$$

Subtracting by parts this equality from the equation (94) for the initial guess $\mathbf{u}^0$ gives

$$\Lambda_E(\mathbf{u}^0 - \mathbf{u}, \mathbf{v}) = \left\langle \left(\mathbf{g}_i - \frac{\mathbf{f}_i}{\omega_q}, \frac{\partial \mathbf{v}}{\partial \xi_i} \right) \right\rangle, \ \mathbf{v} \in \mathbf{Ker}_q.$$

In according to the definition (90) of $\mathbf{g}_i$ in right hand side $\mathbf{g}_i - \mathbf{f}_i/\omega_q \equiv 0$ in $\mathcal{D}_q^{\perp}$ and we should only check, that

$$\int\limits_{\hat{\mathcal{D}}} \left(\mathbf{g}_i - \frac{\mathbf{f}_i}{\omega}, \frac{\partial \mathbf{v}}{\partial \xi_i} \right) d\hat{\mathcal{D}}, \ \mathbf{v} = \mathbf{c}_p, \text{ or } \mathbf{c}_p + C_p \xi \text{ for the elasticity case,}$$

where $\hat{\mathcal{D}}$ is every connected component of the set $\mathbf{T} \setminus \overline{\mathcal{D}_q^{\perp}}$, and $s \times s$ matrix C is independent of ξ and such, that the restriction $C\varepsilon\,|_{\hat{\mathcal{D}}}$ is continuous. Except the elasticity case it is trivial. For the elasticity case it follows from the condition (91). Namely, $\hat{\mathcal{D}}$ consists of one or several domains $\mathcal{D}_p$ and, or $\mathcal{D}_q^{\perp}$ of the original decomposition of the torus $\mathbf{T}$. The restriction $C\varepsilon\,|_{\hat{\mathcal{D}}}$ is continuous, then the restrictions on subdomains of $\hat{\mathcal{D}}$ are continuous as well, and in all these subdomains the condition (91) is valid with this matrix C. That completes the proof with $\omega_q > 0$.

Let now $\omega_q = 0$. Then $\mathbf{u} \in \mathbf{Im}_q$ as an $\mathbf{H}_E$ normal solution, see Lemma 2.1. We verify, that $\mathbf{u}^0 \in \mathbf{Im}_q$,

$$\Lambda_E(\mathbf{u}^0,\mathbf{v}) = \left\langle \left(\mathbf{g}_i, \frac{\partial \mathbf{v}}{\partial \xi_i} \right) \right\rangle = \int\limits_{\mathbf{T}\setminus\overline{\mathcal{D}_q^{\perp}}} \left(\mathbf{g}_i, \frac{\partial \mathbf{v}}{\partial \xi_i} \right) d\xi = 0, \ \mathbf{v} \in \mathbf{Ker}_q.$$

We invoke here the definition (94) of $\mathbf{u}^0$, the condition $\mathbf{g}_i \equiv 0$ in $\mathcal{D}_q^{\perp}$ of (90) and (91).

2. We denote by $\Lambda_{A^0}(\star,\star)$ the bilinear form $\Lambda_A(\star,\star)$ with all $\omega_q = 0$ and consider the representation, cf. (69),

$$\Lambda_A(\mathbf{v},\mathbf{v}) = \Lambda_{A^0}(\mathbf{v},\mathbf{v}) + \sum_q \omega_q \int_{\mathcal{D}_q^\perp} \left(E_{ij}\frac{\partial \mathbf{v}}{\partial \xi_j}, \frac{\partial \mathbf{v}}{\partial \xi_i} \right) d\mathcal{D}_q^\perp.$$

   Lower and upper bounds for the first term in the right hand side were obtained of Lemma 2.2 for $\mathbf{v} \in \mathbf{Im}$. By $0 < \omega_q \le 1 \le \bar{a}_{\mathcal{D}}$, the last term is estimated trivially, as in Lemma 47.

3. By the definition (101) of the operator L, the subspace $\mathbf{Im}_q$ is its invariant subspace iff

$$\Lambda_E(L\mathbf{w},\mathbf{v}) = \Lambda_A(\mathbf{w},\mathbf{v}) = 0, \ \mathbf{w} \in \mathbf{Im}_q, \ \mathbf{v} \in \mathbf{Ker}_q. \tag{104}$$

   But this is just a particular case of the equality (103) for $\mathbf{w} \in \mathbf{Im}_q$. Therefore, all subspaces $\mathbf{Im}_q$ are invariant of the operator L, hence, their intersection $\mathbf{Im} = \cap \mathbf{Im}_q$ is an invariant subspace of L as well.

Π

Proof of Theorem 4.1. We invoke function $\mathbf{u}^0$, a solution of the problem (94) with $\mathbf{g}_i$ of (90) and consider the difference $\mathbf{u}^0 - \mathbf{u}$ of this function and the solution $\mathbf{u}$ of the original problem (85). As in Proof of Theorem 3.1, we have

$$\Lambda_A(\mathbf{u}^0 - \mathbf{u},\mathbf{v}) = \Lambda_A(\mathbf{u}^0,\mathbf{v}) - \left\langle \left(\mathbf{f}_i, \frac{\partial \mathbf{v}}{\partial \xi_i} \right) \right\rangle, \ \mathbf{v} \in \mathbf{H} \tag{105}$$

and present vector $\mathbf{v} \in \mathbf{H}$ here as an orthogonal in $\mathbf{H}_E$ sum,

$$\mathbf{v} = \mathbf{v}_K + \mathbf{v}_I, \ \mathbf{v}_K \in \mathbf{Ker}, \ \mathbf{v}_I \in \mathbf{Im}.$$

We shell now see that all terms with $\mathbf{v}_K$ of (105) vanish. Due to (88) every function $\mathbf{v}_K \in \mathbf{Ker}$ can be written as a sum of functions $\mathbf{v}_q \in \mathbf{Ker}_q$. We use the additive property and find that all terms with $\mathbf{v}_q$ vanish.

For the left hand side, $\Lambda_A(\mathbf{u}^0 - \mathbf{u}, \mathbf{v}_q) = 0$ because of (104), as $\mathbf{u}^0 - \mathbf{u} \in \mathbf{Im} \subseteq \mathbf{Im}_q$ by Lemma 4.1, (102) and $\mathbf{v}_q \in \mathbf{Ker}_q$. In the right hand side both terms are vanished independently by (91). We have to rewrite the first term in according to (103),

$$\Lambda_A(\mathbf{u}^0,\mathbf{v}_q) = \omega_q \Lambda_E(\mathbf{u}^0,\mathbf{v}_q) = \omega_q \left\langle \left(\mathbf{g}_i, \frac{\partial \mathbf{v}_q}{\partial \xi_i} \right) \right\rangle = 0, \ \mathbf{v}_q \in \mathbf{Ker}_q.$$

Therefore, we have just prove that is sufficient to take $\mathbf{v} \in \mathbf{Im}$ in (105).

Much of what follows is the same as of Proof of Theorem 3.1. Π

Proofs of Theorems 4.2 and 4.3 are very similar to the Proofs of Theorem 3.3 and Theorem 3.4 and are not shown here.

5 On a function extension on a torus

5.1

We consider here a possibility of function extension from the set $\mathcal{D} \subset \mathbf{T}$ on the whole torus $\mathbf{T}$. We suppose that the conditions of Subsection 1.7 for $\mathcal{D}$ are fulfilled, i.e., $\mathcal{D}$ is itself a Lipschitz domain, or consists of a finite number of Lipschitz domains $\mathcal{D}_p$ with nonintersecting closures.

We start from the following classical theorem of extensions.[13]

Theorem 5.1 *Let $\Omega \subset \mathbf{R}^\mathbf{s}$ be a bounded Lipschitz domain.*

Then there is a constant $c_1(\Omega) > 0$ such that for a function $v \in W_2^1(\Omega)$ there exists a function $w \in W_2^1(\mathbf{R}^s)$ with finite support such that $w - v \equiv 0$ in Ω and

$$\int\limits_{\mathbf{R}^s} \left(w^2 + \frac{\partial w}{\partial \xi_i} \frac{\partial w}{\partial \xi_i} \right) d\xi \le c_1(\Omega) \int\limits_{\Omega} \left(v^2 + \frac{\partial v}{\partial \xi_i} \frac{\partial v}{\partial \xi_i} \right) d\xi. \tag{106}$$

This theorem leads, in particular, to the equality $W_2^1(\Omega) = W_2^1(\mathbf{R}^s) \mid_\Omega$. Therefore, we can change $v \in W_2^1(\Omega)$ for $v \in W_2^1(\mathbf{R}^s)$ in the formulation of the theorem.

Corollary 5.1 *Let $\Omega \subset \mathbf{R}^\mathbf{s}$ be a bounded Lipschitz domain.*

Then there is a constant $c_1(\Omega) > 0$ such that for a function $v \in W_2^1(\mathbf{R}^s)$ with a finite support there exists a function $w \in W_2^1(\mathbf{R}^s)$ with a finite support such that $w - v \equiv 0$ in Ω and

$$\int\limits_{\mathbf{R}^s} \left(w^2 + \frac{\partial w}{\partial \xi_i} \frac{\partial w}{\partial \xi_i} \right) d\xi \le c_1(\Omega) \int\limits_{\Omega} \left(v^2 + \frac{\partial v}{\partial \xi_i} \frac{\partial v}{\partial \xi_i} \right) d\xi.$$

We note that in the corollary there is a possibility to try the choice $w \equiv v$, because both functions now are of the same functional space, but it is not evident at all that such a choice effects a constant c_1 independent of w.

The next example shows that the statement of the corollary can be break down if the boundary of Ω is not Lipschitz.

Example 5.1 *Let $\Omega = \Omega_0 \cup \Omega_1 \subset \mathbf{R}^2$, where Ω_0 and Ω_1 are unit squares and they have one common vertex only. We consider the functional space $W_2^1(\Omega_0) \times W_2^1(\Omega_1)$, with $W_2^1(\Omega_p) = W_2^1(\mathbf{R}^2) \mid_{\Omega_p}$, $p = 0, 1$. This space is complete, because its multiplicands are complete, as well known. We have $W_2^1(\mathbf{R}^2) \mid_\Omega \subset W_2^1(\Omega_0) \times W_2^1(\Omega_1)$. Let*

$$u \equiv \begin{cases} 0 \text{ in } \Omega_0, \\ 1 \text{ in } \Omega_1 \end{cases}.$$

Then

$$u \in W_2^1(\Omega_0) \times W_2^1(\Omega_1), \text{ but } u \notin W_2^1(\mathbf{R}^2) \mid_\Omega .$$

We now consider any sequence $\{u^n\} \subset W_2^1(\mathbf{R}^2)$ of functions with finite support such that $u^n \mid_\Omega \to u$ in $W_2^1(\Omega_0) \times W_2^1(\Omega_1)$. We suppose that the sequence is fundamental in the space $W_2^1(\mathbf{R}^2)$, then it must converge in $W_2^1(\mathbf{R}^2)$ to a function of $W_2^1(\mathbf{R}^2)$, whose *restriction on Ω have to coincide with u; but, that is impossible. Therefore, the sequence is not fundamental in the space $W_2^1(\mathbf{R}^2)$, i.e., there exist a sequence $\{v^n\}: v^n = u^{i_n} - u^{j_n}$ with $i_n,\ j_n \to \infty$ as $n \to \infty$, that does not tend to zero in the space $W_2^1(\mathbf{R}^2)$.*

On the other hand, the sequence $\{u^n\mid_\Omega\}$ is fundamental in the space $W_2^1(\Omega_0) \times W_2^1(\Omega_1)$ as a converged sequence and, hence, the sequence $\{v^n\mid_\Omega\}$ must tend to zero in the space $W_2^1(\Omega_0) \times W_2^1(\Omega_1)$.

This contradicts to the statement of the Corollary 5.1 for the given Ω with an irregular boundary. This means also, that the set $W_2^1(\mathbf{R}^2)\mid_\Omega$ is not closed in the subspace $W_2^1(\Omega_0) \times W_2^1(\Omega_1)$.

By analogy with Theorem 5.1 and Corollary 5.1 it is possible to prove the following theorem on extension of functions on torus.

Theorem 5.2 *Let Ω and Ω' be bounded Lipschitz domains on torus, $\overline{\Omega} \subset \Omega' \subseteq \mathbf{T}$.*

Then there is a constant $c_2(\Omega,\ \Omega') > 0$ such that for a function $v \in W_2^1(\mathbf{T})$ there exists a function $w \in W_2^1(\mathbf{T})$ with support in Ω' such that $w - v \equiv 0$ in Ω and

$$\int\limits_{\mathbf{T}} \left(w^2 + \frac{\partial w}{\partial \xi_i}\frac{\partial w}{\partial \xi_i} \right) d\mathbf{T} \le c_2(\Omega,\ \Omega') \int\limits_{\Omega} \left(v^2 + \frac{\partial v}{\partial \xi_i}\frac{\partial v}{\partial \xi_i} \right) d\Omega. \tag{107}$$

We have to change here the W_2^1 norm to the W_2^1 seminorm and to consider the case of a nonconnected $\mathcal{D}$ to obtain the next simplest variant of the extension theorem that we use for the diffusion equation.

Theorem 5.3 *Let $\mathcal{D} \subset \mathbf{T}$ be itself a Lipschitz domain, or consists of a finite number of Lipschitz domains $\mathcal{D}_p \subset \mathbf{T}$ with nonintersecting closures.*

Then there is a constant $c_3(\mathcal{D}) > 0$ such that for a function $v \in W_2^1(\mathbf{T})$ there exists a function $w \in W_2^1(\mathbf{T})$ such that $\mathrm{grad}\, w - \mathrm{grad}\, v \equiv 0$ in $\mathcal{D}$ and

$$\int\limits_{\mathbf{T}} \frac{\partial w}{\partial \xi_i}\frac{\partial w}{\partial \xi_i} d\mathbf{T} \le c_3(\mathcal{D}) \int\limits_{\mathcal{D}} \frac{\partial v}{\partial \xi_i}\frac{\partial v}{\partial \xi_i} d\mathcal{D}. \tag{108}$$

Proof of Theorem 5.3. As $\overline{\mathcal{D}_p} \cap \overline{\mathcal{D}_q} = \emptyset,\ p \ne q$, there exists the same number of Lipschitz domains $\mathcal{D}'_p$ such that

$$\overline{\mathcal{D}_p} \subset \mathcal{D}'_p,\ \overline{\mathcal{D}'_p} \cap \overline{\mathcal{D}'_q} = \emptyset,\ p \ne q.$$

For the function $v \in W_2^1(\mathbf{T})$ given let

$$v_p = v - \int\limits_{\mathcal{D}_p} \frac{v\, d\mathcal{D}_p}{mes\ \mathcal{D}_p} \text{ in } \mathcal{D}_p.$$

Evidently, $\int\limits_{\mathcal{D}_p} v_p d\mathcal{D}_p = 0$; then, by Poincare inequality,

$$\int\limits_{\mathcal{D}_p} v_p^2 d\mathcal{D}_p \le c_p \int\limits_{\mathcal{D}_p} \frac{\partial v_p}{\partial \xi_i}\frac{\partial v_p}{\partial \xi_i} d\mathcal{D}_p,$$

therefore

$$\int\limits_{\mathcal{D}_p} \left(v_p^2 + \frac{\partial v_p}{\partial \xi_i} \frac{\partial v_p}{\partial \xi_i} \right) d\mathcal{D}_p \le (1 + c_p) \int\limits_{\mathcal{D}_p} \frac{\partial v_p}{\partial \xi_i} \frac{\partial v_p}{\partial \xi_i} d\mathcal{D}_p.$$

In according to Theorem 5.2 there exist a constant $c_2(\mathcal{D}_p,\ \mathcal{D}'_p) > 0$ and there is a function $w_p \in W_2^1(\mathbf{T})$ with support in $\mathcal{D}'_p$ such that $w_p - v_p \equiv 0$ in $\mathcal{D}_p$ and

$$\int\limits_{\mathcal{D}'_p} \left(w_p^2 + \frac{\partial w_p}{\partial \xi_i} \frac{\partial w_p}{\partial \xi_i} \right) d\mathcal{D}'_p \le$$

$$c_2(\mathcal{D}_p,\ \mathcal{D}'_p) \int\limits_{\mathcal{D}_p} \left(v_p^2 + \frac{\partial v_p}{\partial \xi_i} \frac{\partial v_p}{\partial \xi_i} \right) d\mathcal{D}_p \le$$

$$(1 + c_p) c_2(\mathcal{D}_p,\ \mathcal{D}'_p) \int\limits_{\mathcal{D}_p} \frac{\partial v_p}{\partial \xi_i} \frac{\partial v_p}{\partial \xi_i} d\mathcal{D}_p. \tag{109}$$

We set $w = \sum\limits_p w_p$. In every domain $\mathcal{D}_p$ we have

$$grad\ w = grad\ w_p = grad\ v_p = grad\ v.$$

We conclude that (109) leads to (108) with

$$c_3(\mathcal{D}) = \max_p |(1 + c_p) c_2(\mathcal{D}_p,\ \mathcal{D}'_p)|.$$

Π

We first note that it is possible to add an arbitrary constants to the functions v and w in $\mathbf{T}$, i.e., Theorem 5.3 holds for the factor space $W_2^1(\mathbf{T})/\mathbf{R}$ instead of the ordinary Sobolev space $W_2^1(\mathbf{T})$ as well.

Second, Theorem 5.3 can be easy generalize for vector function by using the component by component extension.

Taking into account an equivalence of the $\mathbf{H}$ and $\mathbf{H}_E$ norms we conclude, except the elasticity case, that Corollary 5.1 on extension holds in the next form

Theorem 5.4 *Let $\mathcal{D} \subset \mathbf{T}$ be itself a Lipschitz domain, or consists of a finite number of Lipschitz domains $\mathcal{D}_p \subset \mathbf{T}$ with nonintersecting closures. Let matrices E_{ij} fulfil the conditions of Subsection 1.5 except the elasticity case.*

There is a constant $\kappa(\mathcal{D}) > 0$ such that for a function $\mathbf{v} \in \{W_2^1(\mathbf{T})\}^m$ there exists a function $\mathbf{w} \in \{W_2^1(\mathbf{T})\}^m$ such that

$$\kappa \int\limits_{\mathbf{T}} \left(E_{ij} \frac{\partial \mathbf{w}}{\partial \xi_i}, \frac{\partial \mathbf{w}}{\partial \xi_i} \right) d\mathbf{T} \le \int\limits_{\mathcal{D}} \left(E_{ij} \frac{\partial \mathbf{v}}{\partial \xi_i}, \frac{\partial \mathbf{v}}{\partial \xi_i} \right) d\mathcal{D} \tag{110}$$

and

$$\frac{\partial v_k}{\partial \xi_i} - \frac{\partial w_k}{\partial \xi_i} \equiv 0 \text{ in } \mathcal{D},\ i = 1, \ldots, s,\ k = 1, \ldots, m. \tag{111}$$

As in the previous scalar functions case, we note again, that it is possible to add an arbitrary independent of ξ vectors to the functions $\mathbf{v}$ and $\mathbf{v}$ in $\mathbf{T}$, i.e., Theorem 5.4 holds for the factor space $\mathbf{H} = \{W_2^1(\mathbf{T})\}^m/\mathbf{R}^m$ instead of the vector Sobolev space $\{W_2^1(\mathbf{T})\}^m$ also.

This comment is still valid for the next theorem, that we state for the elasticity case.

Theorem 5.5 *Let $\mathcal{D} \subset \mathbf{T}$ be itself a Lipschitz domain, or consists of a finite number of Lipschitz domains $\mathcal{D}_p \subset \mathbf{T}$ with nonintersecting closures. Let matrices E_{ij} fulfil the conditions of Subsection 1.5 for the elasticity case.*

There is a constant $\kappa(\mathcal{D}) > 0$ such that for a function $\mathbf{v} \in \{W_2^1(\mathbf{T})\}^m$ there exists a function $\mathbf{w} \in \{W_2^1(\mathbf{T})\}^m$ such that

$$\kappa \int_{\mathbf{T}} \left(E_{ij} \frac{\partial \mathbf{w}}{\partial \xi_i}, \frac{\partial \mathbf{w}}{\partial \xi_i} \right) d\mathbf{T} \leq \int_{\mathcal{D}} \left(E_{ij} \frac{\partial \mathbf{v}}{\partial \xi_i}, \frac{\partial \mathbf{v}}{\partial \xi_i} \right) d\mathcal{D} \tag{112}$$

and

$$\frac{\partial v_i}{\partial \xi_k} + \frac{\partial v_k}{\partial \xi_i} = \frac{\partial w_i}{\partial \xi_k} + \frac{\partial w_k}{\partial \xi_i} \text{ in } \mathcal{D},\ i,\ k = 1, \ldots, s. \tag{113}$$

We note that (110) and (112) are exactly the same.

Our proof of this theorem bases on the next particular case of the known more general statement.[12]

Theorem 5.6 *Let $\Omega \subset \mathbf{R}^s$ be a bounded Lipschitz domain, and $\mathbf{V}_\Omega \subset \{W_2^1(\Omega)\}^s$ is such a subspace, that conditions*

$$\mathbf{v} \in \mathbf{V}_\Omega,\ \varepsilon_{ik}(\mathbf{v}) \stackrel{def}{=} \frac{1}{2} \left(\frac{\partial v_i}{\partial \xi_k} + \frac{\partial v_k}{\partial \xi_i} \right) \equiv 0 \text{ in } \Omega,\ i,\ k = 1, \ldots, s.$$

lead to $\mathbf{v} \equiv 0$ in Ω.

Then there is a constant $c_4(\Omega) > 0$ such that

$$\int_{\Omega} \left[(\mathbf{v}, \mathbf{v}) + \left(\frac{\partial \mathbf{v}}{\partial \xi_i}, \frac{\partial \mathbf{v}}{\partial \xi_i} \right) \right] d\Omega \leq c_4(\Omega) \int_{\Omega} \varepsilon_{ik}(\mathbf{v}) \varepsilon_{ik}(\mathbf{v}) d\Omega,\ \mathbf{v} \in \mathbf{V}|_\Omega\,. \tag{114}$$

Using the same arguments as in the proof of Theorem 5.6,[12] it is easy to prove the next analogous statement for the torus $\mathbf{T}$ instead of $\mathbf{R}^s$.

Corollary 5.2 *Let $\Omega \subset \mathbf{T}$ be a Lipschitz domain, and $\mathbf{V}_\Omega \subset \{W_2^1(\Omega)\}^s$ is such a subspace, that conditions*

$$\mathbf{v} \in \mathbf{V}_\Omega,\ \varepsilon_{ik}(\mathbf{v}) \stackrel{def}{=} \frac{1}{2} \left(\frac{\partial v_i}{\partial \xi_k} + \frac{\partial v_k}{\partial \xi_i} \right) \equiv 0 \text{ in } \Omega,\ i,\ k = 1, \ldots, s.$$

lead to $\mathbf{v} \equiv 0$ in Ω.

Then there is a constant $c_4(\Omega) > 0$ such that

$$\int_{\Omega} \left[(\mathbf{v}, \mathbf{v}) + \left(\frac{\partial \mathbf{v}}{\partial \xi_i}, \frac{\partial \mathbf{v}}{\partial \xi_i} \right) \right] d\Omega \leq c_4(\Omega) \int_{\Omega} \varepsilon_{ik}(\mathbf{v}) \varepsilon_{ik}(\mathbf{v}) d\Omega,\ \mathbf{v} \in \mathbf{V}|_\Omega\,. \tag{115}$$

We now are ready to prove Theorem 5.5.

Proof of Theorem 5.5. We first note, that the condition

$$\varepsilon_{ik}(\mathbf{v}) \stackrel{def}{=} \frac{1}{2}\left(\frac{\partial v_i}{\partial \xi_k} + \frac{\partial v_k}{\partial \xi_i}\right) \equiv 0 \text{ in } \mathcal{D},\ i,\ k = 1,\dots,s, \tag{116}$$

with $\mathbf{v} \in \mathbf{H}$ coincides with the definition of the subspace $\mathbf{Ker}$ of Subsection 2.2, i.e., this condition defines in $\mathbf{H}$ vector functions $\mathbf{w} \in \mathbf{H}$, that must obey the equality $\mathbf{w} = \mathbf{c}_p + C_p\xi$ in $\mathcal{D}_p$ with a vector $\mathbf{c}_p$ and an $s \times s$ matrix $C_p = -C_p^T$, both independent of ξ and restriction $C_p\xi\,|_{\mathcal{D}_p}$ must be continuous in each component of connectedness $\mathcal{D}_p$ of $\mathcal{D}$. The original condition (116) for functions $\mathbf{v} \in \{W_2^1(\mathbf{T})\}^s$ involves also functions $\mathbf{v} \equiv \mathbf{c} \in \mathbf{R}^s$ in $\mathbf{T}$.

Let $\mathbf{V} \subset \{W_2^1(\mathbf{T})\}^s$ be, for example, $\{W_2^1(\mathbf{T})\}^s$ orthogonal complement to the subspace of all functions $\mathbf{v} \in \{W_2^1(\mathbf{T})\}^s$ described by the condition (116), that has been mentioned above. Then every function $\mathbf{v} \in \{W_2^1(\mathbf{T})\}^s$ has a unique representation

$$\mathbf{v} = \mathbf{v}_\mathbf{V} + \mathbf{v}_\mathbf{K},\ \mathbf{v}_\mathbf{V} \in \mathbf{V},\ \mathbf{v}_\mathbf{K} = \mathbf{c}_p + C_p\xi \text{ in } \mathcal{D}_p,$$

with vectors $\mathbf{c}_p$ and $s \times s$ matrices $C_p = -C_p^T$ for every $\mathcal{D}_p$, all independent of ξ and restrictions $C_p\xi\,|_{\mathcal{D}_p}$ must be continuous in each component of connectedness $\mathcal{D}_p$ of $\mathcal{D}$. Then

$$\varepsilon_{ik}(\mathbf{v}) = \varepsilon_{ik}(\mathbf{v}_\mathbf{V}) \text{ in } \mathcal{D}.$$

We apply Theorem 5.6 with $\Omega = \mathcal{D}_p$ and $\mathbf{V}_\Omega = \mathbf{V}\,|_{\mathcal{D}_p}$ for the function $\mathbf{v}_p = \mathbf{v}_\mathbf{V}\,|_{\mathcal{D}_p} \in \{W_2^1(\mathcal{D}_p)\}^s$, and get

$$\int_{\mathcal{D}_p}\left[(\mathbf{v}_p,\mathbf{v}_p) + \left(\frac{\partial \mathbf{v}_p}{\partial \xi_i}, \frac{\partial \mathbf{v}_p}{\partial \xi_i}\right)\right] d\mathcal{D}_p \le c_4(\mathcal{D}_p)\int_{\mathcal{D}_p} \varepsilon_{ik}(\mathbf{v}_p)\varepsilon_{ik}(\mathbf{v}_p)\,d\mathcal{D}_p. \tag{117}$$

Further, we use the vector variant of Theorem 5.2 to extend the function $\mathbf{v}_p$ from $\mathcal{D}_p$ to $\mathbf{T}$ by a function $\mathbf{w}_p \in \{W_2^1(\mathbf{T})\}^s$ such that

$$\mathbf{w}_p - \mathbf{v}_p \equiv \mathbf{0} \text{ in } \mathcal{D}_p,\ supp\,\mathbf{w}_p \subset \mathcal{D}'_p, \text{ and}$$

$$\int_{\mathcal{D}'_p}\left[(\mathbf{w}_p,\mathbf{w}_p) + \left(\frac{\partial \mathbf{w}_p}{\partial \xi_i}, \frac{\partial \mathbf{w}_p}{\partial \xi_i}\right)\right] d\mathcal{D}'_p \tag{118}$$

$$\le c_2(\mathcal{D}_p,\ \mathcal{D}'_p)\int_{\mathcal{D}_p}\left[(\mathbf{v}_p,\mathbf{v}_p) + \left(\frac{\partial \mathbf{v}_p}{\partial \xi_i}, \frac{\partial \mathbf{v}_p}{\partial \xi_i}\right)\right] d\mathcal{D}'_p.$$

Finally, we set

$$\mathbf{w} = \sum_p \mathbf{w}_p \in \{W_2^1(\mathbf{T})\}^s.$$

Then

$$\varepsilon_{ik}(\mathbf{w}) = \varepsilon_{ik}(\mathbf{v}_p) = \varepsilon_{ik}(\mathbf{v}_p) = \varepsilon_{ik}(\mathbf{v}),$$

as in (113). The inequalities (117) and (118) lead to (110) for the particular case of isotropic media. General anisotropic case follows directly from the isotropic one. Π

The last step of this proof leads to a strong dependence of the coefficient κ of matrices E_{ij}. And κ tends to zero when tensor E becomes degenerate or has large coefficients. One practically interest example of large coefficients is the case of almost incompressible media. Using a new approach we proved that it is possible to generalize Theorem 5.5 and other results to such a case.[16] We treated even the limit case of the Stokes' equations for incompressible media as well.

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