Uniform Final Exam for Calculus I – MATH 1401
Spring 2006

Name: ______________________________ ID#: ______________________________

Circle the name of your Recitation Assistant: Ilya Lashuk / Nathan Acks.

Directions:

1. Please PRINT your name and school ID number at the top of this page AND on the next page.
2. Problem #1 is on page 2 and Problem #24 is on page 10.
3. No calculators, computers, books, or external notes. You may use the note sheet provided.
4. Box/circle/highlight your final answers.
5. You may use the back of the sheets as scratch paper, but please indicate clearly where your work is located for each problem.
6. You have three hours to complete the exam. Enjoy.

	Your Score	Possible Points		Your Score	Possible Points
Page 2		21	Page 7		8
Page 3		12	Page 8		9
Page 4		11	Page 9		9
Page 5		9	Page 10		10
Page 6		11	TOTAL		100

Name: ______________________________ ID#: ______________________________

Short Answer. No partial credit. [2 *pts. each*]

(#1) If it's an indefinite integral and you forget the "$+C$", then it's wrong. No negative exponents or fractions in fractions allowed in the final answer.

(a) $[\cos(3x)]' =$

(b) $\int \cos(3x)\,dx =$

(c) In Rolle's Theorem, we have a continuous function f on a closed interval $[a,b]$ and $f(a) = f(b) = 0$. If f is also differentiable on the open interval (a,b), then there exists $c \in (a,b)$ such that something is true. What is it?

(d) $\lim_{x \to +\infty} e^{-x} =$

(e) $\int e^{x/3} dx =$

(f) If $f(x) = |x|$, what is the value of $f'(0)$?

(g) $[\sin^{-1}(2x)]' =$

(h) Recognize? $\int \frac{2x}{3+x^2} dx =$

(i) $\frac{d}{dx}[5^x] =$

Okay, let's get all the limit stuff out of the way...

(#2) [3 *pts.*] Use the LIMIT DEFINITION of $f'(a)$ to find $f'(3)$ if $f(x) = x^2 - x$.

[Do NOT use the Simple Power Rule!]

(#3) [3 *pts. each*] Evaluate the limits.

(a) $\lim_{x \to 0} \frac{e^{2x} - 2x - 1}{x^2}$

(b) $\lim_{x \to +\infty} (1 + x)^{1/x}$

(c) $\lim_{x \to -1^+} \frac{1}{1 + x}$

(#4) [3 *pts.*] Gauss rediscovered these: $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$

Carefully write down and evaluate the Riemann sum associated with trapping the area under the curve $f(x) = x^2 + x$ over the interval $[0, 3]$.

$$\lim_{n \to +\infty} \sum_{k=1}^{n} (width) * f(x_k)$$

Hint: The total width is 3. So the width of rectangle is $\Delta x =$???

[You can check your answer against the value of the definite integral.]

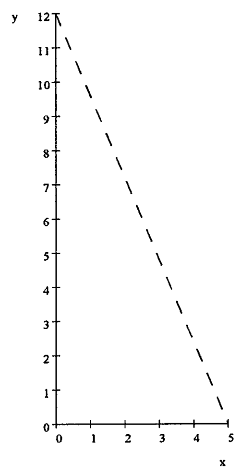
(#5) [4 *pts.*] Falling ladder.

We have a 13-ft ladder leaning against a building. The top of the ladder is presently located at $(0, 12)$ and the bottom of the ladder is located at $(5, 0)$.

This forms a right triangle and, in this case, $y = 12$ and $x = 5$.

Suppose a calculus demon pulls the bottom of the ladder to the right, so that $\dfrac{dx}{dt} = +1$ ft/sec.

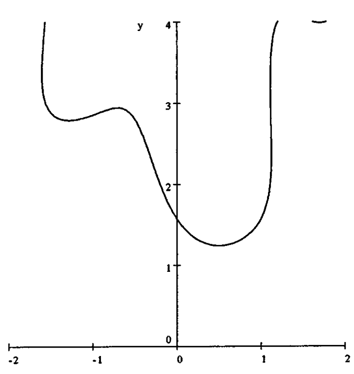
The bottom of the ladder is moving at a constant velocity to the right, and thus, the top of the ladder slides down the side of the building.



Find the value of $\dfrac{dy}{dt}$ when $x = 12$ ft.

(#6) [4 *pts.*] Find the value of y' when $x = 1$ and $y = \dfrac{\pi}{2}$.

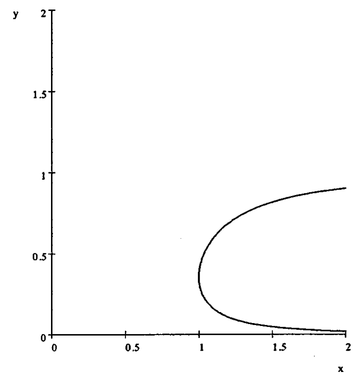
$$\sin(xy) + y - \frac{\pi}{2} = x^2$$



(#7) [3 *pts.*] Simplify as much as possible: $\dfrac{d}{dx}\left[\dfrac{\sqrt{1-x^2}}{x}\right]$

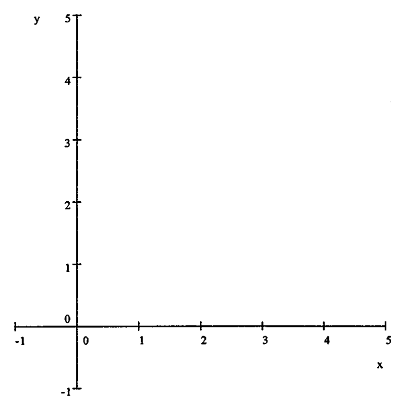
(#8) [3 *pts.*] Find $\frac{dy}{dx}$ if $t = \frac{\pi}{6}$. The parametric curve is defined by:

$$\begin{aligned} x &= \csc(2t) \\ y &= \sin^3(t) \end{aligned}$$



(#9) Let $f(x) = -x^2 + 4x - 3 = -\left(x^2 - 4x + 3\right)$.

(a) [1 *pt.*] Using your knowledge of parabolas, make a reasonable sketch on the axes provided.



(b) [2 *pts.*] Write and evaluate the appropriate definite integral which gives the area trapped *below* the parabola and *above* the x-axis.
(Please do this in the space above!)

(#10) [3 *pts.*] Evaluate:

$$\int_0^{\pi/8} \sec^2(2\theta)\, d\theta$$

(#11) [5 *pts.*] The function below has exactly TWO critical numbers. You know it's the Product Rule and you know that e^{2x} is always positive.

Make a sign chart for $f'(x)$ and then classify the critical numbers as relative min/max or inflection points.

$$f(x) = \left(2x^2 - 6x + 3\right) e^{2x}$$

(#12) [3 *pts.*] Mean Value Theorem.

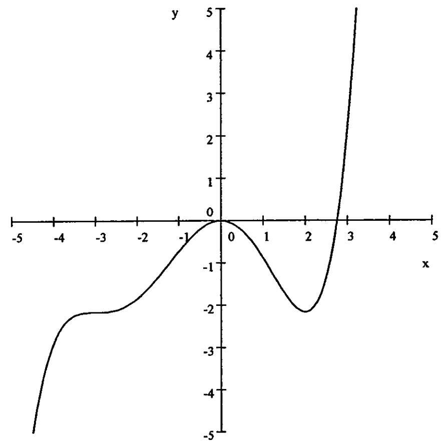
Consider $f(x) = x^3 - x^2$ on the closed interval $[1, 2]$. Find the value of $c \in (1, 2)$ which satisfies the Mean Value Theorem.

Hint: Can't factor it...

(#13) [3 *pts.*] Find all the critical numbers. (No classification necessary.)

$$f(x) = \frac{3}{5}x^{5/3} - 3x^{2/3}$$

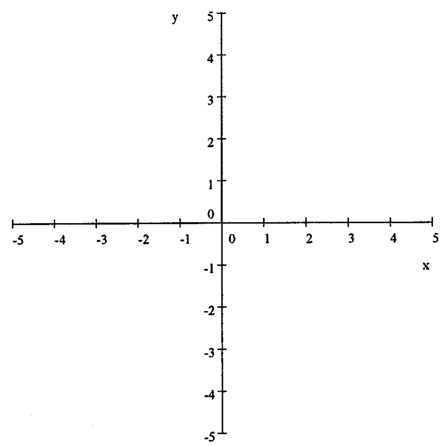
(#14) [4 *pts.*] Here's a graph of $f(x)$. On the axes provided below, make a reasonable sketch of $f'(x)$. Read the hints.



You must declare $x = -3$ critical number!

What are the other critical numbers?

On this graph, point out all inflection points.



On this graph, we need all the important points on the x-axis and the relative extrema! You should point out the corresponding features of this graph with the original!

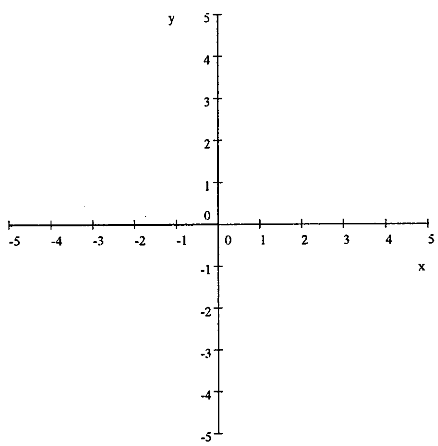
(#15) [4 *pts.*] Consider the function $f(x) = 2x^2 - \sqrt{x}$ on $[0, 1]$.

Find the absolute extrema on this interval.

(#16) [3 *pts.*] An advertisement poster consists of a rectangular printed region plus 1.5-inch margins on the side and 2-inch margins at the top and bottom. If the area of the printed region is to be 300 in^2, find the dimensions of the printed region and the dimensions of the overall poster such that the total area of the poster is MINIMIZED.

[Make our lives easier and let $x =$ the horizontal length of the printed region, and do everything in x.]

(#17) [3 *pts.*] Sketch a CONTINUOUS graph which has the following properties:



(i) $f(0) = 1$

(ii) $f'(0) = 0$

(iii) $f(-2) = 3$

(iii) $f'(3) = 0$

(iv) $f''(x) > 0$ for $-2 < x < 2$

(v) $f''(x) < 0$ for $x > 2$

(vi) $f''(x) > 0$ for $x < -2$

(#18) [3 *pts.*] If $v(t) = 2t + \sin\left(\frac{t}{2}\right)$ is the velocity function, find the distance travelled from $t = 0$ to $t = \frac{\pi}{2}$.

(#19) [3 *pts.*] Find the derivative: $\dfrac{d}{dx}\left[(1+x)^{1/x}\right]$

(#20) [3 *pts.*] Find the linear approximation $L(x)$ for $f(x) = \sqrt{16+x}$ near $x = x_0 = 0$.

(#21) Suppose the position function $s(t) = \dfrac{1}{(t+2)^3}$ is measured in feet and t is measured in seconds with $t \geq 0$.

(a) [2 *pts.*] Find the acceleration function $a(t)$.

(b) [1 *pts.*] What are the units of the acceleration function?

(#22) [3 *pts.*] Find the equation of the tangent line at the point $\left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$.

$$f(x) = \tan^{-1}(2x)$$

(#23) [3 *pts.*] Find the critical number in the interval $[0, \pi]$.

$$f(x) = \sqrt{3}\sin(x) + \cos(x)$$

(#24) [1 *pt. each*] The answer to each of the following is either "Positive", "Negative", or "Zero". For all of these clues, assume that f is a polynomial, and hence all derivatives exist everywhere.

(a) If f is decreasing at $x = a$, then the value of $f'(a)$ is?

(b) If we have a relative minimum at $x = a$, then the value of $f'(a)$ is?

(c) If we have a relative minimum at $x = a$, then the value of $f'(x)$ when x is slightly larger than a is?

(d) If we have an inflection point at $x = a$, then the value of $f''(a)$ is?

Note Sheet for Calculus I Uniform Final – MATH 1401

(#1) Limits. Since you know L'Hôpital's Rule now, you can easily find things like

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1 \quad \text{and} \quad \lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

(a) Remember that $\frac{+\infty}{0^+} \to +\infty$, $\frac{+\infty}{0^-} \to -\infty$, $\frac{0}{\infty} \to 0$, $0^\infty \to 0$, and $(+\infty)^{+\infty} \to +\infty$.

(b) Logarithmic example:

$$\lim_{x \to 0} (1+x)^{1/x} = ???$$

Let $y = \ln\left((1+x)^{1/x}\right) = \frac{1}{x}\ln(1+x) = \frac{\ln(1+x)}{x}$.

$$\lim_{x \to 0} (1+x)^{1/x} = \lim_{x \to 0} e^y = e^{\left(\lim_{x \to 0} y\right)}$$

(#2) Basic Chain Rule forms.

$$[g(x)^n]' = (g(x))^{n-1} g'(x)$$

$$\frac{d}{dx}[\sin(u)] = \cos(u)\frac{du}{dx}$$

$$\frac{d}{dx}[\cos(u)] = -\sin(u)\frac{du}{dx} \quad \text{(Co-functions get the negative!)}$$

$$\frac{d}{dx}[\tan(u)] = \sec^2(u)\frac{du}{dx}$$

$$\frac{d}{dx}[\sec(u)] = \tan(u)\sec(u)\frac{du}{dx}$$

$$\frac{d}{dx}[e^u] = e^u\frac{du}{dx}$$

$$\frac{d}{dx}[b^u] = b^u \ln(b)\frac{du}{dx}, \quad b > 0.$$

$$[\ln(u)]' = \frac{1}{u}\left(\frac{du}{dx}\right)$$

$$\frac{d}{dx}\left[\sin^{-1}(u)\right] = \frac{1}{\sqrt{1-u^2}}\left(\frac{du}{dx}\right)$$

$$\frac{d}{dx}\left[\tan^{-1}(u)\right] = \frac{1}{1+x^2}\left(\frac{du}{dx}\right)$$

(#3) The minimal trig. table...

θ	$\cos(\theta)$	$\sin(\theta)$	$\tan(\theta)$
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{2}$	0	1	$undef.$

(#4) More Derivatives.

(a) The derivative must be a finite number. The derivative at $x = a$ exists if and only if this limit exists:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

This is the slope of the non-vertical tangent line.

(b) Quotient Rule:

$$\left[\frac{f}{g}\right]' = \frac{gf' - fg'}{g^2}.$$

(#5) The Newtonian form of the Chain Rule:

$$[f(g(x))]' = f'(g(x))\, g'(x).$$

The Leibniz form of the Chain Rule:

If the Outer function is $y = f(u)$ and the Inner function is $u = g(x)$, and all function presented are differentiable, then

$$\frac{dy}{dx} = \left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right).$$

(#6) The preferred point-slope form is:

$$y = m(x - x_0) + y_0.$$

The linear approximation of $f(x)$ at a convenient point $x = x_0$ is

$$y = L(x) = f'(x_0)(x - x_0) + f(x_0),$$

which is also a point-slope form.

(#7) A reminder about Implicit Differentiation...

$$\frac{d}{dx}[y^n] = ny^{n-1}y' \quad \text{and} \quad \frac{d}{dt}[\sec(x)] = \tan(x)\sec(x)\left(\frac{dx}{dt}\right)$$

(#8) MVT: If f is cont. on $[a,b]$ and diff. on (a,b), then

$$AROC_{[a,b]} = \frac{f(b) - f(a)}{b - a},$$

and there must exist $c \in (a,b)$ such that...

(#9) Leibniz on parametrics:

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

$$speed(t) = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

(#10) Absolute Extrema: Find, Evaluate, Evaluate, Declare.

Critical numbers: $f'(x) = 0$ or $f'(x)$ is undefined [sharp turn or vertical tangent line]

(#11) $f'(x)$: decreasing/increasing

$f''(x)$: concavity & inflection points.

Inflection point: concavity must change signs!

(#12) Antiderivatives. Since you know the derivatives, you should be able to reverse the process easily.

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1.$$

$$\int \frac{1}{x} dx = \int \frac{dx}{x} = \ln|x| + C$$

$$\int \cos(kx)\, dx = \frac{\sin(kx)}{k} + C$$

$$\int \sin(kx)\, dx = -\frac{\cos(kx)}{x} + C$$ (Co-functions get the negative!)

$$\int \sec^2(kx)\, dx = \frac{\tan(kx)}{k} + C$$

$$\int \tan(kx)\sec(kx)\, dx = \frac{\sec(kx)}{k} + C$$

$$\int \frac{dx}{\sqrt{1-(kx)^2}} = \frac{\sin^{-1}(kx)}{k} + C$$

$$\int \frac{dx}{1+(kx)^2} = \frac{\tan^{-1}(kx)}{k} + C$$

(#13) Recognize:

$$\int \frac{u'(x)}{u} du = \ln |u| + C$$

Example:

$$\int \frac{3x^2}{1+x^3} dx = \int \frac{[1+x^3]'}{1+x^3} dx = \ln \left|1+x^3\right| + C.$$

(#14) Fundamental Theorem of Calculus.

(a) If $F(x)$ is the antiderivative of $f(x)$, then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

(b) The accumulating function (from $x = 0$) is

$$F(x) = \int_0^x f(t)\, dt.$$

This accumulates all the area from $x = 0$ to some other value of x (presumably greater than zero).
Then $F(x)$ is continuous and piecewise differentiable and

$$F'(x) = f(x).$$

Example:

$$F(x) = \int_0^x e^{-t^2} dt$$

There is NO possible elementary antiderivative for this one! But we *can* say that

$$F'(x) = e^{-x^2}.$$

(#15) Solving the integral equation.

If we know that the velocity function is $v(t) = 4t + 6$, and that our initial displacment at $t = 0$ is $s(0) = 8$, we must have

$$s(t) = \int (4t+6)\, dt = 2t^2 + 6t + C.$$

We can find the value of the constant of integration by substituting in the initial conditions.

$$s(0) = 8 = 2\left(0^2\right) + 6(0) + C \Rightarrow C = 8.$$