Review of Material Needed for Math 3191
CU-Denver Mathematics Department

1 Introduction

This is a packet of prerequisite material necessary for understanding material covered in Math 3191: Applied Linear Algebra.

2 Sample Pretest

1. Graph the line $2y - 4x = 8$.
2. What is the domain of the function $f(x) = \sqrt{x-3}$?
3. Calculate the distance between the two points $(1, 2)$ and $(-4, 3)$.
4. Determine where the two lines $2x + y = 4$ and $x + 2y = 5$ intersect.
5. Find all real roots of the polynomial $2x^3 - 3x^2 - 3x + 2$.
6. Let $f(x) = x^2$, $g(x) = x + 1$. Write the composite function $f(g(x))$ in simplest form.
7. Write the following expression as a completely reduced radical expression:
$$(3 + \sqrt{5})^2 - 3(3 + \sqrt{5}) + 2.$$
8. What is $\sin(\pi/3)$?
9. Which of the following statements is equivalent to the statement "If you are taking this course then you have taken Math 1401"?
   (a) If you are not taking this course then you have not taken Math 1401.
   (b) If you have not taken Math 1401 then you are not taking this course.
   (c) If you have taken Math 1401 then you are taking this course.
   (d) If you are not taking this course then you have taken Math 1401.
10. Match each property on the left to a statement on the right that illustrates that property:

(a) Commutativity	(i) $2x + 5y = 5y + 2x$
(b) Associativity	(ii) $6x + 8x = (6 + 8)x$
(c) Additive inverse	(iii) $(3x^3)(1) = 3x^3$
(d) Distributive property	(iv) $5(\pi + (-\pi)) = 5(0)$
(e) Multiplicative identity	(v) $8 + (6 + 5x) = (8 + 6) + 5x$

3 Review Topics

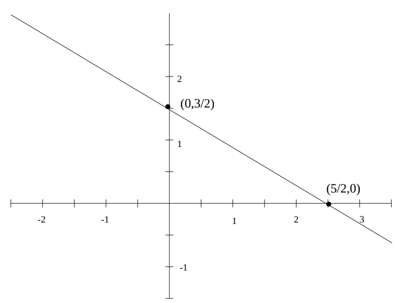
This review covers the following topics, which will be tested on the pretest:

3.1 Graphing Linear Equations

We will do a lot of graphing in this course, so it is important that you are comfortable with this to begin with.

A linear equation is an equation of the form $ax+by=c$, where a, b, and c are constants, and x and y are variables. For example, the equation $3x+\sqrt{2}y=4\pi$ is a linear equation, whereas $3x^2+xy=7$ is not linear. The graph of an equation represents the set of all points that satisfy the equation. The graph of a linear equation is a line.

An easy way to graph a linear equation is to find two points satisfying the equation then draw a line connecting these two points. For example, given the equation $10y+6x=15$, we can find two points satisfying this by 1) setting $y=0$ and solving for x, giving the point $(5/2,0)$, and 2) setting $x=0$, and solving for y, giving the point $(0,3/2)$. Plotting these two points and then drawing a straight line through them gives the desired graph (see figure):



3.2 Domain and Range

The *domain* of a function $f(x)$ is the set of all x for which the function is defined. For example, the domain of the function $f(x)=\sqrt{x}$ is the set of all nonnegative numbers. The *range* of a function is the set of all possible values that the function can evaluate to. For example, the range of $f(x)=\sin x$ is the set $-1 \leq y \leq 1$.

3.3 Distance

The distance between two points (x,y) and (a,b) is given by the formula

$$\text{distance} = \sqrt{(x-a)^2+(y-b)^2}.$$

Example: The distance between the two points $(1,3)$ and $(-2,7)$ is $\sqrt{(1-(-2))^2+(3-7)^2} = \sqrt{3^2+(-4)^2} = \sqrt{25} = 5$.

3.4 Solving systems of equations.

The two most commonly used techniques for solving systems of two equations are as follows. In this course, we will use the second approach, and will generalize it to solve much larger linear systems (involving many equations and many variables).

Approach 1: Solve for one variable in terms of the other. Example: Solve the system of equations

$$\begin{aligned} 2x - 2y^2 &= 4 \\ x - 4y &= -2 \end{aligned}$$

Step 1 Use one of the equations to write one variable as a function of the other one. In our example, we can rewrite the first equation as $x = 2 + y^2$.

Step 2 Substitute into the other equation. In our example, replace x in the second equation by $2 + y^2$, giving the equation $(2 + y^2) - 4y = -2$.

Step 3 Solve for y: $y^2 - 4y + 4 = 0$ or $(y - 2)^2 = 0$. So $y = 2$.

Step 4 Solve for x: $x = 2 + y^2 = 2 + (2)^2 = 6$.

Approach 2: Combine equations .

Step 1 Add a multiple of one equation to the other to eliminate a variable. In our example, we can add (-2) times the second equation to the first, yielding $-2y^2 - (-8y) = 4 - (-4)$ or $-2y^2 + 8y = 8$.

Step 2 Solve for y: We again get $(y - 2)^2 = 0$, so $y = 2$.

Step 3 Solve for x: We now plug $y = 2$ back into the second equation giving $x - 4(2) = -2$, so $x = 6$.

3.5 Finding roots of polynomials.

We will need to be able to find roots of polynomials when we study eigenvalues.

A polynomial is a function of the form $a_n x^n + a_{n-1}x^{(n-1)} + \cdots + a_1 x + a_0$, where n is an integer, and $a_0, a_1, \ldots, a_n$ are constants, with $a_n \neq 0$. The integer n is called the *degree* of the polymomial. The *roots* of the polynomial are the values of x for which the value of the polynomial is 0. For example, 1 is a root of the polyomial $x^3 - 2x + 1$ since $(1)^3 - 2(1) + 1 = 0$.

In order to find the roots of a polynomial, you need to be able to factor the polynomial (that is, you need to rewrite the polynomial as a product of polynomials of lower degree). The roots are then found by setting each factor equal to zero. For example, the polynomial $x^3 - 3x - 2$ can be factored as $(x + 1)^2(x - 2)$. The roots of the polynomial are then given by the setting $x + 1 = 0$ and $x - 2 = 0$, yielding $x = -1, 2$.

Factoring techniques Here are some helpful factoring techniques

1. Find a common factor.

$$\text{Example:}\quad 3x^4 - 12x^2 + 6x = \underbrace{(3x)}_{\text{common factor}}(x^3 - 4x + 2).$$

2. Use a special product:

$$\text{Example:}\quad 4x^2 - 25 = (2x+5)(2x-5).$$

3. Group terms:

$$\text{Example:}\quad \underbrace{x^2 - 2x^2}+\underbrace{3x - 6} = x^2(x-2) + 3(x-2) = (x^2+3)(x-2).$$

4. Look for roots of the form $\pm b/a$, where, a is a factor of the coeficient of the highest order term of the polynomial, and b is a factor of the constant term.

   Example: $2x^2 + 7x + 6$. The coefficient of the highest order term is 2, so we try $a = 1$ or 2. The constant term is 6, so we try $b = 1, 2, 3$ or 6. This gives possible roots of $\pm 1, \pm 1/2, \pm 2, \pm 3, \pm 3/2, \pm 6$. Plugging each of these into the polynomial, we find that $x = -3/2$ and $x = -2$ are both roots. Thus the polynomial has $(x + 3/2)$ and $(x + 2)$ as factors.

3.6 Composition of functions

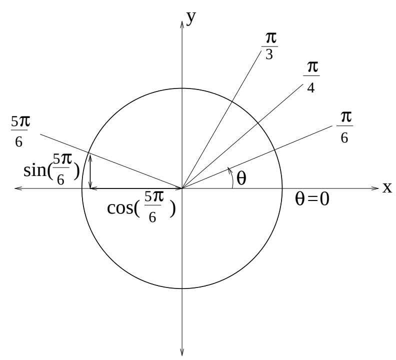
The notion of composition of functions will be used when we study the relationship between linear transformations and matrix multiplication. For functions f and g, the composition of f on g, denoted $f \circ g$, is $(f \circ g)(x) = f(g(x))$, where $g(x)$, where $g(x)$ is in the domain of the function f. The *domain* of $f \circ g$ is the set of input values of x in the domain of g for which $g(x)$ is in the domain of f. For example, let $g(x) = x - 2$, and $f(x) = \sqrt{x}$. Then $(f \circ g)(x) = \sqrt{x-2}$ with domain $x \geq 2$.

3.7 Radical Expressions

Concepts of orthogonality and normalization use the Euclidean definition of distance. Since this involves the square root of a sum of squares, it is essential that you be efficient and comfortble in working with expressions that contain radicals. You should be able to rationalize a radical fraction (that is, remove radicals in the denominator by multiplying both numerator and denominator by the radical so that the radical is eliminated.)

3.8 Trigonometry

We will use some trigonometry in studying linear transformations corresponding to rotations. A very helpful device for remembering your trigonometry is the unit circle shown in the following picture. This is a circle of radius one centered at the origin. To determine the sine and cosine of an angle θ, simply draw a ray from the origin making an angle of θ with the x-axis. The coordinates of the point where this ray intersects the unit circle tell you the sine and cosine of θ. In particular, the x–coordinate is the cosine and the y–coordinate is the sine. In the figure, this is illustrated for the angle $5\pi/6$.



This picture frees you from having to memorize some of the most important trigonometric identities. For example the identity $\sin(\pi - \theta) = \sin(\theta)$ follows from the fact that the rays making angles of θ and $\pi - \theta$ are symmetric about the y-axis. Thus, the points where they intersect the unit circle have the same y–coordinate. In contrast, it is also clear that $\cos(\pi - \theta) = -\cos(\theta)$. Question: how does $\cos(\pi + \theta)$ relate to $\cos(\theta)$?

It will be very helpful to know the sines and cosines of a few key angles: $0, \pi/6\ (30°), \pi/4\ (45°), \pi/3\ (60°), \pi/2\ (90°)$. You should have this memorized! A useful memory device is given by the following chart:

$$\begin{array}{lllll lllll}
\sin(0) & = & \sqrt{\mathbf{0}}/2 & = & 0 & \cos(0) & = & \sqrt{\mathbf{4}}/2 & = & 1 \\
\sin(\pi/6) & = & \sqrt{\mathbf{1}}/2 & = & 1/2 & \cos(\pi/6) & = & \sqrt{\mathbf{3}}/2 & = & \sqrt{3}/2 \\
\sin(\pi/4) & = & \sqrt{\mathbf{2}}/2 & = & \sqrt{2}/2 & \cos(\pi/4) & = & \sqrt{\mathbf{2}}/2 & = & \sqrt{2}/2 \\
\sin(\pi/3) & = & \sqrt{\mathbf{3}}/2 & = & \sqrt{3}/2 & \cos(\pi/3) & = & \sqrt{\mathbf{1}}/2 & = & 1/2 \\
\sin(\pi/2) & = & \sqrt{\mathbf{4}}/2 & = & 1 & \cos(\pi/2) & = & \sqrt{\mathbf{0}}/2 & = & 0 \\
 & & \uparrow & & & & & \uparrow & & \\
 & & \text{see the} & & & & & \text{see the} & & \\
 & & \text{pattern?} & & & & & \text{pattern?} & &
\end{array}$$

3.9 Logic

The study of linear algebra involves proofs. Because of this, it is important that you understand some basic concepts from logic.

Propositions: A **proposition** is a statement that is either true or false, but not both. For example, the following are propositions:

1. Denver is the capital of Colorado.
2. $1 + 1 = 3$.
3. Someone in this room is 32 years old.

The first proposition is true, the second is false, and the third is either true or false, but we don't know which.

In the study of logic, letters (such as $p, q, r, s, \ldots$) are used to represent propositions. To each proposition, we can assign a **truth value**, denoted T if the proposition is true and F if the proposition is false.

Logical Operators: Many mathematical statements are constructed by combining one or more propositions using logical operators. Consider the following two statements:

- p: Today is Friday.
- q: It is raining.

The following table defines the main logical operators:

Logical Operator	informal name	notation	example
negation	not	$\neg p$	Today is *not* Friday
conjunction	and	$p \wedge q$	Today is Friday *and* it is raining.
disjunction	or	$p \vee q$	Today is Friday *or* it is raining. (or both)
exclusive or	xor	$p \oplus q$	Today is Friday *or* it is raining *but not both.*
implication	if/then	$p \rightarrow q$	*If* today is Friday *then* it is raining.

Truth Tables: In order to determine whether a compound statement is true or false, it is helpful to define a **truth table** for each of the above logical operator. A truth table displays the relationships between the truth values of the original propositions and the truth values of the compound proposition. Each row of the truth table gives one possible combination of the truth values of the statements.

Truth table for negation

p	$\neg p$
T	F
F	T

This truth table illustrates that if the negation $\neg p$ will always have the opposite truth value as p. In other words, if the statement "Today is Friday" is true, then the negation "Today is not Friday" must be false (and vice versa).

Truth tables for the other operators are more complicated because they dependent on every possible combination of truth values for p and q.

Conjunction (AND)		
p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction (OR)		
p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The above truth tables illustrate that a conjunction will be true only if both proposition p is true *and* proposition q is true. In contrast, the disjunction is true if either proposition p is true or proposition q is true. The exclusive-or of two statements is true if either one or the other but not both is true. This is illustrated in the truth table for exclusive-or.

Exclusive Or		
p	q	$p \wedge q$
T	T	F
T	F	T
F	T	T
F	F	F

Implication (If/Then)		
p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

The truth table for implications can be a little confusing, so let us examine it more carefully. According to the truth table, the implication $p \to q$ is false only if p is false and q is true. So for example, in a strictly logical sense, the following statement is true: "If $1 + 1 = 3$ then I will win the lottery today." This may be unsettling since the value of $1 + 1$ really has nothing to do with the lottery. However, since $1 + 1 = 3$ is false, the compound statement is true regardless of whether or not I will win the lottery. The reason we define truth values for implications this way becomes clearer when we consider how implications are most often used. Many important theorems in mathematics are stated in the form of an implication. For example the statement "All differentiable functions are continuous" is an implication. To see this, observe that we could rewrite it in the form "If f is a differentiable function, then f is continuous". Given a theorem of the form $p \to q$, the truth table shows us that if p is true, then q must be true (it is the only possible option in the truth table). However, if p is false, then we cannot determine whether or not q is true. For example, if f is a differentiable function, the above theorem tells us that f must be continuous. However, if f is not differentiable, we cannot determine from the theorem whether or not f is continuous. Based on this discussion, it is perhaps clearer to write the truth table in the following form:

Implication (If/Then)		
p	$p \to q$	q
T	T	T
T	F	F
F	T	don't know

Because implications arise in many places in mathematical reasoning, a wide variety of terminology is used to express $p \to q$. Here are some of the more commonly used ways:

- if p then q
- p implies q
- if p, q
- p is sufficient for q

Converse and Contrapositive: For an implication $p \to q$, there are some related implications that are of particular interest. The **converse** is the statement $q \to p$. The **contrapositive** is the statement $\neg q \to \neg p$. For example, consider the statement "If today is Thursday, then I have a test today." The converse of this statement is "If I have a test today, then today is Thursday." The contrapositve is "If today is not Thursday, then I do not have a test today."

Using a truth table, we can examine how each of these implications compares to the original implication:

Related Implications				
p	q	$p \to q$	$q \to p$	$\neg q \to \neg p$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Observe that the contrapositive always has the same truth values as the original implication. In other words, the statement "If today is not Thursday, then I do not have a test today" is logically equivalent to the statement "If today is Thursday, then I have a test today". This fact is the basis for many mathematical proofs:

One way to prove the theorem $p \to q$ is to prove the contrapositive $\neg q \to \neg p$.

For example, one way to prove the theorem "if f is differentiable then it is continuous" would be to show that if f is not continuous, it cannot be differentiable.

3.10 Properties of multiplication and addition of Real Numbers

You should be familiar with the following properties of multiplication and addition of real numbers. We will expand on these concepts quite a bit when we study vector spaces, so it is vital that you have already had some introduction to these concepts.

commutativity $a + b = b + a$, $ab = ba$. (Notice that this property does not hold for division: $a/b \neq b/a$.

associativity $(a+b)+c = a+(b+c)$. (Again, notice that this property doesn't hold for divisions).

distributive property $a(b + c) = ab + bc$.

additive inverse $a + (-a) = 0$

additive and multiplicative identities $a + 0 = a$, $a(1) = a$.