University of Colorado at Denver — Mathematics Department

Applied Linear Algebra Preliminary Exam

January 12, 2007, 10:00 am – 2:00 pm

Name: ____

The proctor will let you read the following conditions before the exam begins, and you will have time for questions. Once the exam begins, you will have 4 hours to do your best. This is a closed book exam. Please put your name on each sheet of paper that you turn in.

Exam conditions:

- Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write legibly using a dark pencil or pen.
- Notation: $\mathcal{C}$ denotes the field of complex numbers , $\mathcal{R}$ denotes the field of real numbers, and F denotes a field which may be either $\mathcal{C}$ or $\mathcal{R}$. $\mathcal{C}^n$ and $\mathcal{R}^n$ denote the vector spaces of n-tuples of complex and real scalars, respectively. T^* is the adjoint of the operator T and λ^* is the complex conjugate of the scalar λ. v^T and A^T denote vector and matrix transposes, respectively.
- Ask the proctor if you have any questions.

Good luck!

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________	8.	________

Total ________

1. LetV be the usual real vector space of all 3×3 real matrices. Define subsets of V by

$$U = \{A \in V : A^T = A\} \text{ and } W = \{A \in V : A^T = -A\}.$$

(a) (3 points) Show that both U and W are subspaces of V.

(b) (3 points) Show that $V = U \oplus W$.

(c) (3 points) Determine the dimensions of U and W.

(d) (3 points) Let $Tr : V \to \mathcal{R} : A \mapsto \text{trace}(A)$. Put $R = \{A \in U : Tr(A) = 0\}$. Show that R is a subspace of U and compute a basis for R.

(e) (8 points) Put $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ and define $T : W \to W : B \mapsto PB + BP^T$.

Show that $T \in \mathcal{L}(W)$ and determine a basis $\mathcal{B}$ for W. Then compute the matrix $[T]_{\mathcal{B}}$. What are the minimal and characteristic polynomials for T?

Sketch of solution to part (e): Put

$$\mathbf{w}_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \mathbf{w}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}; \quad \mathbf{w}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Then $\mathcal{B} = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ is a basis for W and

$$T : \mathbf{w}_1 \mapsto -\mathbf{w}_3; \quad T : \mathbf{w}_2 \mapsto \mathbf{w}_1; \quad T : \mathbf{w}_3 \mapsto \mathbf{w}_2.$$

It follows that

$$[T]_{\mathbf{B}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}.$$

It is easy to compute that $P^3 = -I$. So the minimal polynomial for T divides $p(x) = x^3 + 1 = (x+1)(x^2 - x + 1)$, with $x^2 - x + 1$ irreducible over $\mathcal{R}$. Also, $P^2 - P + I \neq 0$ and $P \neq -I$, so $p(x)$ is the minimal (and characteristic) polynomial for T.

2. (20 points) Let V be a finite dimensional inner product space over $\mathcal{C}$. Suppose that T is a positive operator on V (called *positive semidefinite* by some authors). Prove that T is invertible if and only if

$$\langle Tv, v \rangle > 0$$

for every $v \in V \setminus \{0\}$.

Solution: Since V is finite dimensional, T is invertible if and only if $T(v) \neq 0$ for all nonzero vectors v. Clearly if $\langle Tv, v \rangle > 0$ for every $v \in V \setminus \{0\}$, then $T(v) \neq 0$ for all nonzero vectors v and T is invertible. Conversely, suppose $\langle Tv, v\rangle = 0$ for some nonzero vector v. Since T is a positive operator, there is an operator S for which $T = S^*S$. Then $0 = \langle Tv, v\rangle = \langle S^*Sv, v\rangle = \langle Sv, Sv\rangle$. This implies $Sv = 0$, and in turn $Tv = S^*Sv = 0$, implying T is not invertible.

3. (20 points) Suppose $T \in \mathcal{L}(V)$ is invertible. Prove that there exists a polynomial $p \in F[x]$ such that $T^{-1} = p(T)$.

Solution: Since T is invertible it does not have 0 as an eigenvalue. Hence its minimal polynomial q does not have t as a factor. Thus, the minimal polynomial has the form

$$q(t) = a_0 + a_1 t + \cdots a_k t^k,$$

for some $k \geq 1$, with $a_k \neq 0$ and $a_0 \neq 0$. Define $b_i = a_i/a_0$, for $i = 1, \ldots, k$, and define $p(t) = -b_1 - b_2 t - \cdots - b_k t^{k-1}$. Then

$$0 = q(T)/a_0 = I - p(T)T \Longrightarrow I = p(T)T.$$

Thus, $T^{-1} = p(T)$.

4. Let U and W be subspaces of the finite-dimensional inner product space V.

(a) (10 points) Prove that $U^\perp \cap W^\perp = (U + W)^\perp$.

(b) (10 points) Prove that

$$\dim(W) - \dim(U \cap W) = \dim(U^\perp) - \dim(U^\perp \cap W^\perp).$$

Solution: Let $x \in U^\perp \cap W^\perp$ be arbitrary. Then for any $u \in U$ and $w \in W$, $\langle x, u + w\rangle = \langle x, u\rangle + \langle x, w\rangle = 0$. Thus, $x \in (U + W)^\perp$, so $U^\perp \cap W^\perp \subset (U + W)^\perp$.

For any $y \in (U + W)^\perp$, and any $u \in U$ and $w \in W$, we have $u = u + 0 \in U + W$, so $\langle y, u\rangle = 0$. Similarly, $\langle y, w\rangle = 0$. Thus, $y \in U^\perp \cap W^\perp$. Thus, $(U + W)^\perp \subset U^\perp \cap W^\perp$. It follows that $(U + W)^\perp = U^\perp \cap W^\perp$, proving part (a).

Keep in mind that for finite-dimensional inner product spaces we know that $\dim(U^\perp) = \dim(V) - \dim(U)$. Then for the proof of (b) consider the following:

$$\begin{aligned}
\dim(U^\perp) &- \dim(U^\perp \cap W^\perp) = (\dim(V) - \dim(U)) - \dim\left((U + W)^\perp\right) \\
&= \dim(V) - \dim(U) - (\dim(V) - \dim(U + W)) \\
&= \dim(U) + \dim(W) - \dim(U \cap W) - \dim(U) \\
&= \dim(W) - \dim(U \cap W), \quad \text{as desired.}
\end{aligned}$$

5. Let B be an $n \times n$ (complex) hermitian matrix. Let λ_1 be the maximum of the norms of the eigenvalues of B. For $\bar{0} \neq \mathbf{x} \in \mathcal{C}^n$, and using the usual 2-norm $\|\mathbf{x}\| = \|\mathbf{x}\|_2$, define the Rayleigh Quotient $\rho_B(\mathbf{x})$ for B by

$$\rho_B(\mathbf{x}) = \frac{\langle B\mathbf{x}, \mathbf{x}\rangle}{\langle \mathbf{x}, \mathbf{x}\rangle} = \frac{\mathbf{x}^* B\mathbf{x}}{\|\mathbf{x}\|^2}.$$

Prove the following:

(i) (10 points) If B is $n \times n$ hermitian with λ_1 defined as above, prove that

$$\lambda_1 = \max\{\rho_B(\mathbf{x}) : \mathbf{x} \in \mathcal{C}^n \text{ and } \|\mathbf{x}\| = 1\}.$$

(ii) (10 points) Let A be any $n \times n$ complex matrix with largest singular value σ_1. If $\|A\|_2 = \max\{\|A\mathbf{x}\| : \mathbf{x} \in \mathcal{C}^n \text{ and } \|\mathbf{x}\| = 1\}$, show that

$$\|A\|_2 = \sigma_1.$$

Solution: First note that if $0 \neq k \in \mathcal{C}$ and $\bar{0} \neq \mathbf{x} \in \mathcal{C}^n$, then $\rho_B(k\mathbf{x}) = \rho_B(\mathbf{x})$. If we put $\mathcal{O} = \{\mathbf{x} \in \mathcal{C}^n : \|\mathbf{x}\| = 1\}$, then

$$\sup\{\rho_B(\mathbf{x}) : \bar{0} \neq \mathbf{x} \in \mathcal{C}^n\} = \sup\{\rho_B(\mathbf{x}) : \mathbf{x} \in \mathcal{O}\}.$$

Second, since B is hermitian, B has real eigenvalues which we may order as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Moreover, there is an orthonormal basis $\mathcal{B} = (v_1, \ldots, v_n)$ of eigenvectors so that $Bv_j = \lambda_j v_j$, for $j = 1, 2, \ldots, n$. If we put v_j in as the jth column of the $n \times n$ matrix P, then P is unitary ($P^* = P^{-1}$) and $P^*BP = \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$. Since $\mathbf{y} \mapsto P\mathbf{y} = \mathbf{x}$ maps $\mathcal{O}$ to $\mathcal{O}$ in a one-to-one and onto manner, we have

$$\begin{aligned}
\sup\{\rho_B(\mathbf{x}) : \mathbf{x} \in \mathcal{O}\} &= \sup\{\mathbf{x}^* B\mathbf{x} : \mathbf{x} \in \mathcal{O}\} \\
= \sup\{(P\mathbf{y})^* B(P\mathbf{y}) : \mathbf{y} \in \mathcal{O}\} &= \sup\{\mathbf{y}^* \Lambda \mathbf{y} : \mathbf{y} \in \mathcal{O}\} \\
&= \sup\{\sum_{j=1}^{n} \lambda_j |y_j|^2 : (y_1, \ldots, y_n)^T \in \mathcal{O}\} \\
&\leq \sup\{\lambda_1 \sum_{j=1}^{n} |y_j|^2 : \sum_{j=1}^{n} |y_j|^2 = 1\} = \lambda_1.
\end{aligned}$$

So to prove part (i), we just need to find an $\mathbf{x} \in \mathcal{O}$ for which $\rho_B(\mathbf{x}) = \lambda_1$. Clearly $\mathbf{x} = v_1$ will work (with $\mathbf{y} = P^{-1}\mathbf{x} = (1, 0, \ldots, 0)^T$).

For part (ii), we note that $B = A^*A$ is hermitian, and we can adapt the notation of part (i) and use the fact that the largest eigenvalue of A^*A is $\lambda_1 = \sigma_1^2$ to obtain

$$\begin{aligned}
\|A\|_2 &= \max\{\|A\mathbf{x}\| : \mathbf{x} \in \mathcal{C}^n \text{ and } \|\mathbf{x}\| = 1\} \\
&= \max\{\sqrt{\mathbf{x}^* A^* A\mathbf{x}} : \mathbf{x} \in \mathcal{O}\} \\
&= \sqrt{\sigma_1^2} = \sigma_1. \text{(By part (i))}
\end{aligned}$$

6. (20 points) Let W be the space of 3×2 matrices over $\mathcal{R}$. Put $V_1 = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \\ c & 0 \end{pmatrix} : a, b, c \in \mathcal{R} \right\}$,

and $V_2 = \left\{ \begin{pmatrix} 0 & a \\ 0 & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathcal{R} \right\}$. So W is isomorphic to $V_1 \oplus V_2$.

Let A be the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \\ 3 & 4 & -1 \end{pmatrix}$$

Define a map $T \in \mathcal{L}(W)$ by $T : W \rightarrow W : B \mapsto AB$.

Compute the eigenvalues of T, the minimal polynomial for T, and the characteristic polynomial for T. Compute the Jordan form for T. (Hint: One eigenvector of the matrix A is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.)

Solution: Since the subspaces V_1 and V_2 of W are invariant under T, and T essentially acts the same on V_1 as it does on V_2, we first compute the minimal polynomial of $T|_{V_i}$ (T restricted to V_i), where i is either 1 or 2. Here the minimal poly equals the characteristic poly equals

$$\det(xI - A) = (x-6)(x^2+6x+6) = (x-6)(x-(-3+\sqrt{3}))(x-(-3-\sqrt{3})).$$

So on W, T has minimal poly equal to $p(x) = (x-6)(x^2+6x+6)$ and characteristic polynomial equal to $c_T(x) = p(x)^2$. It follows that T is diagonalizable with Jordan form the 6×6 diagonal matrix with each of its three distinct eigenvalues (i.e., the roots of $p(x) = 0$) appearing twice.

7. Let U and V be inner product spaces over the field F.

   (i) (3 points) Complete the following definition: A *norm* on the vector space V is a function

   $$\| \ \| : V \rightarrow [0, \infty) \subseteq \mathcal{R} \text{ such that } \ldots$$

   Solution: (i) $\|v\| = 0$ iff $v = \overline{0}$.
   (ii) $\|av\| = |a| \cdot \|v\|$ for all $a \in F$ and all $v \in V$.
   (iii) $\|u+v\| \leq \|u\| + \|v\|$ for all $u, v \in V$.

   (ii) (3 points) Define the three norms (1-norm, 2-norm and ∞-norm) often used on finite dimensional vector spaces over $\mathcal{C}$.

Solution:

$$\|\mathbf{x}\|_1 = |x_1| + |x_2| + \cdots + |x_n|; \quad \text{(the 1-norm)} \tag{1}$$

$$|\mathbf{x}\|_2 = (|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2)^{1/2}; \quad \text{(the 2-norm)} \tag{2}$$

$$\|\mathbf{x}\|_\infty = \max\{|x_1|, |x|_2, \ldots, |x_n|\}. \quad \text{(the } \infty - \text{norm)} \tag{3}$$

(iii) (4 points) Suppose that U and V are endowed with norms $\| \quad \|_U$ and $\| \quad \|_V$, respectively. Complete the following definition:

For each $T \in \mathcal{L}(U, V)$ a *transformation norm* $\|T\|_{U,V}$ may be defined as follows:

Solution:

$$\|T\|_{U,V} = \max\{\|T(u)\|_V / \|u\|_U : \bar{0} \neq u \in U\} = \max\{\|T(u)\|_V : \|u\|_U = 1\}.$$

(iv) (10 points) Let A be an $m \times n$ matrix over $\mathcal{C}$. On the one hand we may view A as an element of an mn-dimensional vector space over $\mathcal{C}$. So A may be given one of the vector norms described in parts (i) and (ii). On the other hand, if $\mathcal{S}_1$ is the standard basis for $\mathcal{C}^n$ with the standard inner product and $\mathcal{S}_2$ is the standard basis for $\mathcal{C}^m$ with the standard inner product, then A may be viewed as the matrix representing the linear map $T_A \in \mathcal{L}(\mathcal{C}^n, \mathcal{C}^m)$ defined by

$$T_A : \mathbf{x} \mapsto A\mathbf{x}.$$

Then if T_A has one of the transformation norms of part (iii), A may be given the transformation norm of T_A. Suppose that $\mathcal{C}^n$ and $\mathcal{C}^m$ are given their respective 1-norms. Show that

$$\|A\|_1 = \alpha := \max\{\sum_{i=1}^m |A_{ij}| : 1 \leq j \leq n\}.$$

Solution: Recall that $\|\mathbf{x}\| = |x_1| + |x_2| + \cdots + |x_n|$, so that

$$\|A\|_1 = \max\left\{\frac{\|A\mathbf{x}\|_1}{\|\mathbf{x}\|_1} : \bar{0} \neq \mathbf{x} \in \mathcal{C}^n\right\}.$$

Suppose that $\max\{\sum_{i=1}^m |A_{ij}| : 1 \leq j \leq n\}$ occurs when $j = j_0$, so $\alpha = \sum_{i=1}^m |A_{ij_0}|$. Then observe

$$\begin{aligned}
\|A\mathbf{x}\|_1 &= \sum_{i=1}^m \left|\sum_{j=1}^n A_{ij}x_j\right| \leq \sum_{i=1}^m \sum_{j=1}^n |A_{ij}| \cdot |x_j| \\
&= \sum_{j=1}^n \left(\sum_{i=1}^m |A_{ij}|\right) |x_j| \leq \sum_{j=1}^n \left(\max_j \sum_{i=1}^m |A_{ij}|\right) |x_j| \\
&= \left(\max_j \sum_{i=1}^m |A_{ij}|\right) \|x\|_1 = \alpha \|\mathbf{x}\|_1.
\end{aligned}$$

This says that $\frac{\|A\mathbf{x}\|_1}{\|\mathbf{x}\|_1} \le \alpha$, so $\|A\|_1 \le \alpha$. To complete the proof, we need to construct a vector $\mathbf{x} \in F^n$ such that $\|A\mathbf{x}\|_1 = \alpha\|\mathbf{x}\|_1$. Put $\mathbf{x} = (0, 0, \ldots, 1, \ldots, 0)^T$ where the single nonzero entry 1 is in the j_0 spot. Then $\|\mathbf{x}\|_1 = 1$, and

$$\|A\mathbf{x}\|_1 = \sum_{i=1}^{m} |A_{ij_0}| = \alpha = \alpha\|\mathbf{x}\|_1, \quad \text{as desired.}$$

8. (a) (7 points) Show that if $C \in M_n(\mathcal{C})$ (the set of $n \times n$ matrices over $\mathcal{C}$) is Hermitian and $x^*Cx = 0$ for all $x \in \mathcal{C}^n$, then $C = 0$.

   (b) (6 points) Show that for any $A \in M_n(\mathcal{C})$ there are (unique) Hermitian matrices B and C for which $A = B + iC$.

   (c) (7 points) Show that if x^*Ax is real for all $x \in \mathcal{C}^n$, then A is Hermitian.

Solution: (a) First we give an elegant solution. There is a unitary matrix U for which $C = U\Lambda U^*$, with $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of C. So $U^*CU = \Lambda$. Use the hypothesis with x equal to the jth column of U to force $\lambda_j = 0$ for all $j = 1, \ldots, n$. Hence $C = 0$.

For a more elementary solution, let x be the vector with a 1 in position j and 0 elsewhere to force $c_{jj} = 0$. Then for $1 \le k < j \le n$, Let x have a 1 in position k, a 1 in position j, and 0 elsewhere to force $\bar{c}_{kj} + c_{kj} = 0$. Then let x have a 1 in position k, an i (the complex number) in position j, and 0 elsewhere to force $-\bar{c}_{kj} + c_{kj} = 0$. These last two equalities force $c_{kj} = 0$. Hence $C = 0$.

(b) If we assume $A = B + iC$ with $B = B^*$ and $C = C^*$, then $A^* = B^* - iC^* = B - iC$. Hence $B = (A + A^*)/2$ and $C = (A - A^*)/(2i)$. It is easy to check that both B and C are Hermitian.

(c) Suppose x^*Ax is real for all $x \in \mathcal{C}^n$. From part (b) we can write $A = B + iC$ with B and C both Hermitian. So $x^*Ax = x^*Bx + ix^*Cx$ must be real for all $x \in \mathcal{C}^n$. Also, x^*Bx and x^*Cx must be real since B and C are Hermitian. Hence x^*Cx must be 0 for all $x \in \mathcal{C}^n$, forcing $C = 0$ by part (a). Hence $A = B$ is Hermitian.