**University of Colorado at Denver — Mathematics Department**

**Applied Linear Algebra Preliminary Exam**

**January 12, 2007, 10:00 am – 2:00 pm**

Name: ______________________________________________

The proctor will let you read the following conditions before the exam begins, and you will have time for questions. Once the exam begins, you will have 4 hours to do your best. This is a closed book exam. Please put your name on each sheet of paper that you turn in.

**Exam conditions**:

- Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write legibly using a dark pencil or pen.
- Notation: $\mathcal{C}$ denotes the field of complex numbers , $\mathcal{R}$ denotes the field of real numbers, and $F$ denotes a field which may be either $\mathcal{C}$ or $\mathcal{R}$. $\mathcal{C}^n$ and $\mathcal{R}^n$ denote the vector spaces of $n$-tuples of complex and real scalars, respectively. $T^*$ is the adjoint of the operator $T$ and $\lambda^*$ is the complex conjugate of the scalar $\lambda$. $v^T$ and $A^T$ denote vector and matrix transposes, respectively.
- Ask the proctor if you have any questions.

Good luck!

| | | | |
|---|---|---|---|
| 1. | ________ | 5. | ________ |
| 2. | ________ | 6. | ________ |
| 3. | ________ | 7. | ________ |
| 4. | ________ | 8. | ________ |
| | | Total | ________ |

1. Let$V$ be the usual real vector space of all $3 \times 3$ real matrices. Define subsets of $V$ by

$$U = \{A \in V : A^T = A\} \text{ and } W = \{A \in V : A^T = -A\}.$$

   (a) (3 points) Show that both $U$ and $W$ are subspaces of $V$.

   (b) (3 points) Show that $V = U \oplus W$.

   (c) (3 points) Determine the dimensions of $U$ and $W$.

   (d) (3 points) Let $Tr : V \to \mathcal{R} : A \mapsto \text{trace}(A)$. Put $R = \{A \in U : Tr(A) = 0\}$. Show that $R$ is a subspace of $U$ and compute a basis for $R$.

   (e) (8 points) Put $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ and define $T : W \to W : B \mapsto PB - BP^T$. Show that $T \in \mathcal{L}(W)$ and determine a basis $\mathcal{B}$ for $W$. Then compute the matrix $[T]_{\mathcal{B}}$. What are the minimal and characteristic polynomials for $T$?

2. (20 points) Let $V$ be a finite dimensional inner product space over $\mathcal{C}$. Suppose that $T$ is a positive operator on $V$ (called *positive semidefinite* by some authors). Prove that $T$ is invertible if and only if

$$\langle Tv, v\rangle \; > \; 0$$

   for every $v \in V \setminus \{0\}$.

3. (20 points) Suppose $T \in \mathcal{L}(V)$ is invertible. Prove that there exists a polynomial $p \in F[x]$ such that $T^{-1} = p(T)$.

4. Let $U$ and $W$ be subspaces of the finite-dimensional inner product space $V$.

   (a) (10 points) Prove that $U^\perp \cap W^\perp = (U + W)^\perp$.

   (b) (10 points) Prove that

$$\dim(W) - \dim(U \cap W) = \dim(U^\perp) - \dim(U^\perp \cap W^\perp).$$

5. Let $B$ be an $n \times n$ (complex) hermitian matrix. Let $\lambda_1$ be the maximum of the norms of the eigenvalues of $B$. For $\overline{0} \neq \mathbf{x} \in \mathcal{C}^n$, and using the usual 2-norm $\|\mathbf{x}\| = \|\mathbf{x}\|_2$, define the Rayleigh Quotient $\rho_B(\mathbf{x})$ for $B$ by

$$\rho_B(\mathbf{x}) = \frac{\langle B\mathbf{x}, \mathbf{x}\rangle}{\langle \mathbf{x}, \mathbf{x}\rangle} = \frac{\mathbf{x}^* B\mathbf{x}}{\|\mathbf{x}\|^2}.$$

   Prove the following:

   (i) (10 points) If $B$ is $n \times n$ hermitian with $\lambda_1$ defined as above, prove that

$$\lambda_1 = \max\{\rho_B(\mathbf{x}) : \mathbf{x} \in \mathcal{C}^n \text{ and } \|\mathbf{x}\| = 1\}.$$

(ii) (10 points) Let $A$ be any $n \times n$ complex matrix with largest singular value $\sigma_1$. If $\|A\|_2 = \max\{\|A\mathbf{x}\| : \mathbf{x} \in \mathcal{C}^n \text{ and } \|\mathbf{x}\| = 1\}$, show that

$$\|A\|_2 = \sigma_1.$$

6. (20 points) Let $W$ be the space of $3\times2$ matrices over $\mathcal{R}$. Put $V_1 = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \\ c & 0 \end{pmatrix} : a, b, c \in \mathcal{R} \right\}$,

and $V_2 = \left\{ \begin{pmatrix} 0 & a \\ 0 & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathcal{R} \right\}$. So $W$ is isomorphic to $V_1 \oplus V_2$.

Let $A$ be the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \\ 3 & 4 & -1 \end{pmatrix}$$

Define a map $T \in \mathcal{L}(W)$ by $T : W \to W : B \mapsto AB$.

Compute the eigenvalues of $T$, the minimal polynomial for $T$, and the characteristic polynomial for $T$. Compute the Jordan form for $T$. (Hint: One eigenvector of the matrix $A$ is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.)

7. Let $U$ and $V$ be inner product spaces over the field $F$.

(i) (3 points) Complete the following definition: A *norm* on the vector space $V$ is a function

$$\| \ \| : V \to [0, \infty) \subseteq \mathcal{R} \text{ such that } \ldots$$

(ii) (3 points) Define the three norms (1-norm, 2-norm and $\infty$-norm) often used on finite dimensional vector spaces over $\mathcal{C}$.

(iii) (4 points) Suppose that $U$ and $V$ are endowed with norms $\| \ \|_U$ and $\| \ \|_V$, respectively. Complete the following definition:

For each $T \in \mathcal{L}(U, V)$ a *transformation norm* $\|T\|_{U,V}$ may be defined as follows:

(iv) (10 points) Let $A$ be an $m \times n$ matrix over $\mathcal{C}$. On the one hand we may view $A$ as an element of an $mn$-dimensional vector space over $\mathcal{C}$. So $A$ may be given one of the vector norms described in parts (i) and (ii). On the other hand, if $\mathcal{S}_1$ is the standard basis for $\mathcal{C}^n$ with the standard inner product and $\mathcal{S}_2$ is the standard basis for $\mathcal{C}^m$ with the standard inner product, then $A$ may be viewed as the matrix representing the linear map $T_A \in \mathcal{L}(\mathcal{C}^n, \mathcal{C}^m)$ defined by

$$T_A : \mathbf{x} \mapsto A\mathbf{x}.$$

Then if $T_A$ has one of the transformation norms of part (iii), $A$ may be given the transformation norm of $T_A$. Suppose that $\mathcal{C}^n$ and $\mathcal{C}^m$ are given their respective 1-norms. Show that

$$\|A\|_1 = \alpha := \max\{\sum_{i=1}^{m} |A_{ij}| : 1 \leq j \leq n\}.$$

8. (a) (7 points) Show that if $C \in M_n(\mathcal{C})$ (the set of $n \times n$ matrices over $\mathcal{C}$) is Hermitian and $x^*Cx = 0$ for all $x \in \mathcal{C}^n$, then $C = 0$.

   (b) (6 points) Show that for any $A \in M_n(\mathcal{C})$ there are (unique) Hermitian matrices $B$ and $C$ for which $A = B + iC$.

   (c) (7 points) Show that if $x^*Ax$ is real for all $x \in \mathcal{C}^n$, then $A$ is Hermitian.