**UCDHSC — Dept. of Mathematical Sciences**

**Applied Linear Algebra Preliminary Exam**

**(Solutions) May 25, 2007**

1. For each of the following statements, prove that it is true or give an example to show that it is false. Throughout, $A \in \mathcal{C}^{m\times m}$ unless otherwise indicated.

   (a) If $\lambda$ is an eigenvalue of $A$ and $\mu \in \mathcal{C}$, then $\lambda - \mu$ is an eigenvalue of $A - \mu I$.

   (b) If $A$ is real and $\lambda$ is an eigenvalue of $A$, then so is $-\lambda$.

   (c) If $A$ is real and $\lambda$ is an eigenvalue of $A$, then so is $\bar{\lambda}$.

   (d) If $\lambda$ is an eigenvalue of $A$ and $A$ is nonsingular, then $\lambda^{-1}$ is an eigenvalue of $A^{-1}$.

   (e) If all the eigenvalues of $A$ are zero, then $A = 0$.

   (f) If $A$ is Hermitian and $\lambda$ is an eigenvalue of $A$, then $|\lambda|$ is a singular value of $A$.

   (g) If $A$ is diagonalizable and its eigenvalues are equal, then $A$ is diagonal.

   **Solution.**

   (a) True. If $\lambda$ is an eigenvalue of $A$, then there exists $x$ different from 0 such that $Ax = x\lambda$ which implies that $(A - \mu I)x = x(\lambda - \mu)$ and so $\lambda - \mu$ is an eigenvalue of $A - \mu I$.

   (b) False. Take the identity matrix as a counter-example. All the eigenvalues are 1s.

   (c) True. If $A$ is real then the characteristic polynomial of $A$ ($p_A(x) = \det(xI - A)$) has real coefficients. If $\lambda$ is an eigenvalue of $A$ then it is a root of the characteristic polynomial. Since the coefficients of $p_A(x)$ are all real, its roots are complex conjugates, that is to say if $\lambda$ is a root of $p_A(x)$, then so is $\bar{\lambda}$ (possibly the same root if $\lambda$ is real). Since the roots of $p_A(x)$ are eigenvalues of $A$, we conclude that $\bar{\lambda}$ is an eigenvalue of $A$.

   (d) True. First note that if $A$ is nonsingular then $A^{-1}$ exists and we also have that all the eigenvalues of $A$ are nonzero so that $\lambda^{-1}$ is well defined. If $\lambda$ is an eigenvalue of $A$, then there exists $x$ different from 0 such that $Ax = x\lambda$ which implies that $A^{-1}x = x\lambda^{-1}$ so $\lambda^{-1}$ is an eigenvalue of $A^{-1}$.

   (e) False. Take
   $$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$
   the only eigenvalue of $A$ is 0 but $A$ is not 0.

   (f) True. There is two ways to go.

      i. If $\lambda$ is an eigenvalue of $A$, then, first, note that $\lambda$ is real (since $A$ is Hermitian), second, there exists $x$ of norm 1 such that $Ax = x\lambda$ which implies that
      $$Ax = (x\,\mathrm{sign}(\lambda))\,|\lambda|,$$
      where $\mathrm{sign}(\lambda) = 1$ if $\lambda \geq 0$ and $\mathrm{sign}(\lambda) = -1$ otherwise.
      Let us call $v = x$, $\sigma = |\lambda|$, and $u = (x\,\mathrm{sign}(\lambda))$, then this rewrites
      $$Av = u\sigma. \quad (*)$$

Since $A$ is Hermitian, $A^*x = x\lambda$ which implies that

$$A^* \left(x \operatorname{sign}(\lambda)\right) = x|\lambda|.$$

So,

$$A^* u = v\sigma. \quad (**)$$

Equalities $(*)$ and $(**)$ imply that $(u, \sigma, v)$ is a singular triplet for $A$. We conclude that, if $A$ is Hermitian and $\lambda$ is an eigenvalue of $A$, then $|\lambda|$ is a singular value of $A$.

ii. If $A$ is Hermitian then $A$ is unitarily diagonalizable; that is to say: $A = V\Lambda V^*$, where $\Lambda$ is a diagonal real matrix with the eigenvalues of $A$ on the diagonal and $V$ is unitary ($VV^* = I$). Consider the diagonal matrix, $D$, made of $+1$ and $-1$ on the diagonal defined as $d_{ii} = \operatorname{sign}(\lambda_i)$, then note that:

$$D\Lambda = |D| \quad \text{and} \quad D^2 = I$$

Defining $\Sigma = |D| = D\Lambda$ and $U = VD$, we can write

$$A = V\Lambda V^* = (VD)(D\Lambda)V^* = U|\Lambda|V^* = U\Sigma V^*.$$

Note that $U$ and $V$ are unitary, $\Sigma$ is diagonal with positive real elements on the diagonal so that $A = U\Sigma V^*$ is a singular value decomposition of $A$. Since we have defined $\Sigma = |\Lambda|$, we conclude that, if $A$ is Hermitian and $\lambda$ is an eigenvalue of $A$, then $|\lambda|$ is a singular value of $A$.

(g) True. If $A$ is diagonalizable and its eigenvalues are equal, then there exists a nonsingular matrix $V$ such that $A = VDV^{-1}$ with $D = \lambda I$, so $A = V(\lambda I)V^{-1} = \lambda I$, thus $A$ is diagonal. ($A$ is called a homothecy.)

2. Suppose $A, B \in \mathcal{C}^{m\times m}$ such that $\operatorname{Spectrum}(A) \cap \operatorname{Spectrum}(B) = \emptyset$.

(a) Let $p_A(x)$ denote the characteristic polynomial of $A$. Prove that $p_A(B)$ is nonsingular.

(b) Consider $X \in \mathcal{C}^{m\times m}$, such that $AX = XB$. Prove that for any polynomial $P$, we have $P(A)X = XP(B)$.

(c) Prove that

$$\forall X \in \mathcal{C}^{m\times m}, (AX = XB \Longrightarrow X = 0).$$

(d) Prove that $\forall C \in \mathcal{C}^{m\times m}$, there exists a unique $X \in \mathcal{C}^{m\times m}$, such that $AX - XB = C$.

**Solution.**

(a) Using the Fundamental Theorem of Algebra, there exists $\lambda_1$, $\lambda_2$, …, $\lambda_m$ in $\mathcal{C}$ such that

$$p_A(x) = \prod_{i=1}^{m} (\lambda_i - x),$$

therefore

$$p_A(B) = \prod_{i=1}^{m} (\lambda_i I - B).$$

$$\begin{aligned} p_A(B) \text{ is nonsingular} \quad &\Longleftrightarrow \quad (\forall i \in \{1,\dots,n\}, \lambda_i I - B \text{ is nonsingular }),\\ &\Longleftrightarrow \quad (\forall i \in \{1,\dots,n\}, \lambda_i \notin \text{ Spectrum}(B)),\\ &\Longleftrightarrow \quad \text{Spectrum}(A) \cap \text{Spectrum}(B) = \emptyset. \end{aligned}$$

(b) Let us consider $X \in \mathcal{C}^{m\times m}$, such that $AX = XB$, then

$$A^2X = A(AX) = A(XB) = (AX)B = (XB)B = XB^2.$$

We can show (by induction) that, for any positive integer $k$, we have

$$A^kX = XB^k.$$

Therefore, for any polynomial $P$, we have

$$P(A)X = XP(B).$$

(c) If we consider the previous equation with the characteristic polynomial of $A$, this gives us

$$p_A(A)X = Xp_A(B).$$

But $p_A(A) = 0$ (Cayley-Hamilton Theorem), and thus

$$Xp_A(B) = 0.$$

We have proved in (a) that $p_A(B)$ is nonsingular, therefore

$$X = 0.$$

(d) The mapping

$$\begin{aligned} \phi: \quad \mathcal{C}^{m\times m} &\longrightarrow \mathcal{C}^{m\times m}\\ X &\longmapsto AX - XB \end{aligned}$$

is linear. It is injective (see (c.)). Since $\mathcal{C}^{m\times m}$ is of finite dimension, $\phi$ is bijective.

Remark: the equation $AX - XB = C$ is called the (continuous) Sylvester equation.

3. A matrix $A \in \mathcal{C}^{n\times n}$ is tripotent if $A^3 = A$. Show that every tripotent matrix is diagonalizable. (Hint: Observe that $A$ is annihilated by the polynomial $p(t) = t^3 - t$.)

   **Solution:** Observe that $A$ is annihilated by the polynomial $p(t) = t^3 - t = t(t-1)(t+1)$. The minimal polynomial divides this polynomial, so is of the form $m(t) = t^{d_0}(t-1)^{d_1}(t+1)^{d_2}$, with $0 \le d_i \le 1$ for $i = 0, 1, 2$. Thus, each Jordan block in the Jordan canonical form has order 1, which means that the Jordan canonical form is diagonal, so $A$ is diagonalizable.

4. Let $V$ be a finite-dimensional vector space over the field $\mathbb{F}$. Suppose $T \in \mathcal{L}(V)$ is self-adjoint, $\lambda \in \mathbb{F}$, and $\epsilon > 0$. Suppose there exists $v \in V$ such that $\|v\| = 1$ and

$$\|Tv - \lambda v\| < \epsilon.$$

Prove that $T$ has an eigenvalue $\lambda'$ such that $|\lambda - \lambda'| < \epsilon$.

**Solution:** Since $T$ is self-adjoint, it has (by the spectral theorem) a set of orthonormal eigenvectors $(e_1, \ldots, e_n)$ that form a basis for $V$. Let $\lambda_1, \ldots, \lambda_n$ be the corresponding eigenvalues. Let $v$ be as described above, and let $(v_1, \ldots, v_n)$ be its coordinates with respect to the basis. That is,

$$v = v_1 e_1 + \cdots + v_n e_n.$$

Then $Tv = \lambda_1 v_1 e_1 + \cdots + \lambda_n v_n e_n$. Since $\|Tv - \lambda v\| < \epsilon$, we have

$$\begin{aligned}
\epsilon^2 &> \|Tv - \lambda v\|^2 \\
&= \|(\lambda_1 - \lambda) v_1 e_1 + \cdots + (\lambda_n - \lambda) v_n e_n\|^2 \\
&= |\lambda_1 - \lambda|^2 |v_1|^2 + \cdots |\lambda_n - \lambda|^2 |v_n|^2 \\
&\geq \left(\min\{|\lambda_1 - \lambda|^2, \cdots, |\lambda_n - \lambda|^2\}\right) \left(|v_1|^2 + \cdots + |v_n|^2\right) \\
&= \left(\min\{|\lambda_1 - \lambda|^2, \cdots, |\lambda_n - \lambda|^2\}\right) \|v\|^2 \\
&= \left(\min\{|\lambda_1 - \lambda|^2, \cdots, |\lambda_n - \lambda|^2\}\right). \\
&= |\lambda_j - \lambda|^2 \text{ for some } j.
\end{aligned}$$

Thus, $\epsilon > |\lambda_j - \lambda|$ for some eigenvalue $\lambda_j$.

5. Compute the singular value decomposition of $A = \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & -i \end{pmatrix}$.

**Solution:** $AA^* = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. The eigenvalues of this matrix are $\lambda_1 = 3$ and $\lambda_2 = 1$. The corresponding unit eigenvectors are $u_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$, and $u_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$. The singular values of $A$ are $s_1 = \sqrt{\lambda_1} = \sqrt{3}$ and $s_2 = \sqrt{\lambda_2} = 1$. Thus,

$$A = U \Sigma V^*,$$

where $U = (u_1 \;\; u_2)$, $\Sigma = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, and $V^*$ is a $3 \times 3$ unitary matrix, with columns $v_1, v_2$ and $v_3$. Then

$$v_1 = \frac{1}{s_1} A^* u_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ -\frac{2i}{\sqrt{6}} \end{pmatrix}.$$

and

$$v_2 = \frac{1}{s_2} A^* u_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}.$$

$v_3$ must be a unit vector orthogonal to both $v_1$ and $v_2$. Thus, $\langle (1,-1,-2i)^T, v_3 \rangle = 0$, and $\langle (1,1,0)^T, v_3 \rangle = 0$. It follows that $v_3 = \left( \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{i}{\sqrt{3}} \right)^T$, so

$$V = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ -\frac{2i}{\sqrt{6}} & 0 & \frac{i}{\sqrt{3}} \end{pmatrix},$$

and the singular value decomposition of $A$ is

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2i}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{i}{\sqrt{3}} \end{pmatrix}.$$

6. Let $V$ be a vector space over the field $\mathbb{F}$, and let $T$ be a linear map from $V$ to $\mathbb{F}$. Prove that if $u \in V$ is not in $\operatorname{null} T$, then

$$V = \operatorname{null} T \oplus \{au : a \in \mathbb{F}\}.$$

**Solution:** Suppose $u \in V$ is not in T. If $a \in \mathbb{F}$ and $au \in \operatorname{null} T$, then $0 = T(au) = aTu$, which implies that $a = 0$ (because $Tu \neq 0$). Thus

$$\operatorname{null} T \bigcap \{au : a \in \mathbb{F}\} = \{0\}.$$

If $v \in V$, then find $a$ such that $v = (v - au) + au$ and $v - au \in \operatorname{null} T$. To find such an $a$, note that $T(v - au) = 0 \Leftrightarrow a = \frac{Tv}{Tu}$. Thus,

$$v = \left( v - \frac{Tv}{Tu} u \right) + \frac{Tv}{Tu} u,$$

which expresses $v$ as the sum of a vector in $\operatorname{null} T$ and a scalar multiple of $u$. Hence $V = \operatorname{null} T + \{au : a \in \mathbb{F}\}$. Since $\operatorname{null} T \bigcap \{au : a \in \mathbb{F}\} = \{0\}$, we have $V = \operatorname{null} T \oplus \{au : a \in \mathbb{F}\}$.

7. Prove that if there exists a linear map on $V$ whose null space and range are both finite dimensional, then $V$ is finite dimensional.

**Solution:** Suppose there is a linear map $T : V \to W$ such that $\operatorname{null} T$ and $\operatorname{range} T$ are both finite dimensional. Then there exist vectors $u_1, \ldots, u_m \in V$ and $w_1, \ldots, w_n \in \operatorname{range} T$ such that $(u_1, \ldots, u_m)$ spans $\operatorname{null} T$ and $(w_1, \ldots, w_n)$ spans $\operatorname{range} T$. Because each $w_j \in \operatorname{range} T$, there exits $v_j \in V$ such that $w_j = Tv_j$.

Suppose $v \in V$. Then $Tv \in \operatorname{range} T$, so there exist $b_1, \ldots, b_n \in \mathbb{F}$ such that

$$\begin{aligned} Tv &= b_1 w_1 + \cdots + b_n w_n \\ &= b_1 T v_1 + \cdots + b_n T v_n \\ &= T(b_1 v_1 + \cdots + b_n v_n) \end{aligned}$$

It follows that $v - (b_1v_1 + \cdots + b_nv_n) \in \operatorname{null} T$. Thus, there exist $a_1, \ldots, a_m$ such that

$$v - (b_1v_1 + \cdots + b_nv_n) = a_1u_1 + \cdots + a_mu_m,$$

or

$$v = b_1v_1 + \cdots + b_nv_n + a_1u_1 + \cdots + a_mu_m.$$

So, $(v_1, \ldots, v_n, u_1, \ldots, u_m)$ spans $V$, so $V$ is finite-dimensional.

8. Let $\beta = \{v_1, \ldots, v_5\}$ be an arbitrary ordered basis for $\Re^5$. Let $A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 3 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

   Define $T : \Re^5 \to \Re^5$ by $[T(v)]_\beta = A[v]_\beta$ for all $v \in \Re^5$. (Note: the notation $[w]_\beta$ denotes the coordinate representation of $w$ relative to the basis $\beta$.)

   (a) Compute the eigenvalues of $T$ and both the minimal and characteristic polynomials of $T$.

   (b) Find the Jordan form for $T$.

**Solution:**

(a) Since $A$ is an upper triangular matrix, the eigenvalues of $T$ lie on the diagonal of $A$. Thus, $T$ has eigenvalues $\lambda = 1$ with algebraic multiplicity 3 and $\lambda = 2$ with algebraic multiplicity 2. The characteristic polynomial is

$$p(x) = (x-1)^3(x-2)^2.$$

The minimium polynomial $m(x)$ must divide this polynomial, so

$$m(x) = (x-1)^{d_1}(x-2)^{d_2},$$

where $1 \le d_1 \le 3$ and $1 \le d_2 \le 2$. Now,

$$A - I = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

which has a null space of dimension 2. Thus, $\lambda = 1$ has two ordinary eigenvectors corresponding to $\lambda = 1$, which forces $d_1 = 2$. Similarly,

$$A - 2I = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

which has a null space of dimension 1. Thus, $\lambda = 2$ has one ordinary eigenvalue, which forces $d_2 = 2$. Thus, the minimal polynomial is

$$m(x) = (x-1)^2(x-2)^2.$$

(b) Using the information from above, the Jordan form is

$$J = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$