

University of Colorado at Denver — Mathematics Department

Applied Linear Algebra Preliminary Exam

May 25, 2007

Name: _____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value unless otherwise specified.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write legibly using a dark pencil or pen.
- Notation: \mathfrak{R} denotes the set of real numbers; \mathcal{C} denotes the set of complex numbers; \mathbb{Z} denotes the set of integers; \mathbb{Q} denotes the set of rational numbers; and \mathbb{F} denotes an arbitrary field. These extend to vector spaces as \mathfrak{R}^n , \mathcal{C}^n , \mathbb{Z}^n , \mathbb{Q}^n and \mathbb{F}^n , respectively. $\mathcal{L}(V)$ denotes the set of linear operators on the vector space V . Other notation will be defined as needed.
- Ask the proctor if you have any questions.

Good luck!

- | | |
|----------|----------|
| 1. _____ | 5. _____ |
| 2. _____ | 6. _____ |
| 3. _____ | 7. _____ |
| 4. _____ | 8. _____ |

Total _____

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

1. For each of the following statements, prove that it is true or give an example to show that it is false. Throughout, $A \in \mathcal{C}^{m \times m}$ unless otherwise indicated.
 - (a) If λ is an eigenvalue of A and $\mu \in \mathcal{C}$, then $\lambda - \mu$ is an eigenvalue of $A - \mu I$.
 - (b) If A is real and λ is an eigenvalue of A , then so is $-\lambda$.
 - (c) If A is real and λ is an eigenvalue of A , then so is $\bar{\lambda}$.
 - (d) If λ is an eigenvalue of A and A is nonsingular, then λ^{-1} is an eigenvalue of A^{-1} .
 - (e) If all the eigenvalues of A are zero, then $A = 0$.
 - (f) If A is Hermitian and λ is an eigenvalue of A , then $|\lambda|$ is a singular value of A .
 - (g) If A is diagonalizable and its eigenvalues are equal, then A is diagonal.

2. Suppose $A, B \in \mathcal{C}^{m \times m}$ such that $\text{Spectrum}(A) \cap \text{Spectrum}(B) = \emptyset$.

(a) Let $p_A(x)$ denote the characteristic polynomial of A . Prove that $p_A(B)$ is nonsingular.

(b) Consider $X \in \mathcal{C}^{m \times m}$, such that $AX = XB$. Prove that for any polynomial P , we have $P(A)X = XP(B)$.

(c) Prove that

$$\forall X \in \mathcal{C}^{m \times m}, (AX = XB \implies X = 0).$$

(d) Prove that $\forall C \in \mathcal{C}^{m \times m}$, there exists a unique $X \in \mathcal{C}^{m \times m}$, such that $AX - XB = C$.

3. A matrix $A \in \mathcal{C}^{n \times n}$ is tripotent if $A^3 = A$. Show that every tripotent matrix is diagonalizable. (Hint: Observe that A is annihilated by the polynomial $p(t) = t^3 - t$.)

4. Let V be a finite-dimensional vector space over the field \mathbb{F} . Suppose $T \in \mathcal{L}(V)$ is self-adjoint, $\lambda \in \mathbb{F}$, and $\epsilon > 0$. Suppose there exists $v \in V$ such that $\|v\| = 1$ and

$$\|Tv - \lambda v\| < \epsilon.$$

Prove that T has an eigenvalue λ' such that $|\lambda - \lambda'| < \epsilon$.

5. Compute the singular value decomposition of $A = \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & -i \end{pmatrix}$.

6. Let V be a vector space over the field \mathbb{F} , and let T be a linear map from V to \mathbb{F} . Prove that if $u \in V$ is not in $\text{null } T$, then

$$V = \text{null } T \oplus \{au : a \in \mathbb{F}\}.$$

7. Prove that if there exists a linear map on V whose null space and range are both finite dimensional, then V is finite dimensional.

8. Let $\beta = \{v_1, \dots, v_5\}$ be an arbitrary ordered basis for \mathfrak{R}^5 . Let $A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 3 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

Define $T : \mathfrak{R}^5 \rightarrow \mathfrak{R}^5$ by $[T(v)]_\beta = A[v]_\beta$ for all $v \in \mathfrak{R}^5$. (Note: the notation $[w]_\beta$ denotes the coordinate representation of w relative to the basis β .)

- (a) Compute the eigenvalues of T and both the minimal and characteristic polynomials of T .
- (b) Find the Jordan form for T .