University of Colorado at Denver — Mathematics Department

Applied Linear Algebra Preliminary Exam

January 13, 2006

Name: ______________________________

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value unless otherwise specified.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write legibly using a dark pencil or pen.
- Notation: $\Re$ denotes the set of real numbers; $\mathcal{C}$ denotes the set of complex numbers; $\mathbb{Z}$ denotes the set of integers; and, $\mathbb{Q}$ denotes the set of rational numbers. These extend to vector spaces as $\Re^n$, $\mathcal{C}^n$, $\mathbb{Z}^n$, and $\mathbb{Q}^n$, respectively. $\mathcal{L}(V)$ denotes the set of linear operators on V. Other notation will be defined as needed.
- Ask the proctor if you have any questions.

Good luck!

1. __________
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5. __________
6. __________
7. __________
8. __________

Total __________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

1. Let
$$A = \begin{bmatrix} -10 & 36 & 36 \\ -3 & 11 & 9 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 9 & 5 & -6 \\ 9 & 3 & -4 \end{bmatrix}.$$
Are A and B similar matrices? Why or why not?

   Solution: The characteristic polynomial of A is $(2-\lambda)\left((-10-\lambda)(11-\lambda)+108\right) = (2-\lambda)(-2-\lambda+\lambda^2) = (2-\lambda)^2(-1-\lambda)$. Thus, A has eigenvalues $-1, 2, 2$. Solving $(A-2I)x = 0$ reveals that the eigenspace for $\lambda = 2$ has dimension 2. Thus, A has a complete set of linearly independent eigenvectors, so is diagonalizable. Thus, A is similar to the diagonal matrix $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

   Similarly, the characteristic polynomial of B is $(2-\lambda)\left((5-\lambda)(-4-\lambda)+18)\right) = (2-\lambda)^2(-1-\lambda)$. So B has the same eigenvalues as A, and the eigenspace of $\lambda = 2$ also has dimension 2. Thus, B is similar to D. Since A and B are similar to the same diagonal matrix D, they are similar.

2. Let V be a complex inner product space. For $T \in \mathcal{L}(V)$, define $H = \frac{1}{2}(T + T^*)$ and $S = \frac{1}{2}(T - T^*)$. Prove that T is normal if every eigenvector of H is also an eigenvector of S.

   Solution: The key theorem needed for this problem is that an operator T on a complex inner-product space V is normal if and only if V has an orthonormal basis consisting of eigenvectors of T.

   H is clearly self-adjoint (and therefore normal), so it has an orthonormal set $\mathcal{B}$ of eigenvectors that form a basis for V. If every eigenvector of H is an eigenvector of S, then for each $v \in \mathcal{B}$,
$$Tv = (H + S)v = (\lambda + \mu)v,$$
   where λ and μ are the eigenvalues associated with v for H and S, respectively. It follows that all the vectors in the orthonormal basis $\mathcal{B}$ are eigenvectors of T, so by the theorem stated above, T is normal.

3. Let A be an $m \times n$ real matrix and $b \in \Re^m$. Show that exactly one of the following systems has a solution:

   i) $Ax = b$
   ii) $A^T y = 0, \quad y^T b \neq 0.$

   Note: Our notation is $y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$, so $y^T = [y_1, \ldots, y_m]$.

Solution: If $b \in \operatorname{col} A$, then statement i) has a solution, but since $\operatorname{col} A \perp \operatorname{null} A^T$, statement ii) has no solution.

If $b \notin \operatorname{col} A$, then statement i) does not have a solution. In this case, let $z = \operatorname{proj}_{\operatorname{col} A} b$ (the orthogonal projection of b onto the column space of A), and define $y = b - z$. Note that $y \neq 0$ (since $b \notin \operatorname{col} A$). Note also that since z is an orthogonal projection, $y \in (\operatorname{col} A)^{\perp} = \operatorname{null} A^T$. Thus, $A^T y = 0$ and $y^T b = y^T(y+z) = y^T y \neq 0$, so statement ii) has a solution.

4. Consider the matrix $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. Let $b \in \Re^2$ be a unit vector and let $\delta b \in \Re^2$ be a small perturbation vector, with $\|\delta b\| = \epsilon > 0$. Let x solve $Ax = b$, and let δx be the error associated with the perturbation δb, which satisfies $A(x+\delta x) = b + \delta b$.

   (a) Calculate the condition number of A.

   (b) Determine the least upper bound on the relative error $\frac{\|\delta x\|}{\|x\|}$

   (c) Find a b and δb for which this upper bound is achieved (subject to $\|b\| = 1$ and $\|\delta b\| = \epsilon$).

Solution:

(a) The eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 1$. Since A is symmetric, $\|A\| = \max\{|\lambda_1|, |\lambda_2|\} = 3$ and $\left\|A^{-1}\right\| = \max\left\{|1/\lambda_1|, |1/\lambda_2|\right\} = 1$. Thus, $c(A) = \|A\| \left\|A^{-1}\right\| = 3$.

(b)

$$\frac{\|\delta x\|}{\|x\|} \leq c(A)\frac{\|\delta b\|}{\|b\|} = 3\epsilon.$$

(c) Observe that $\delta x = A^{-1}\delta b$, and $x = A^{-1}b$. The relative error is maximized by choosing the unit vector b that minimizes $\|\delta x\|$ and by choosing the perturbation vector δb that maximizes $\|\delta x\|$, subject to $\|\delta b\| = \epsilon$. This is accomplished by choosing b to be an eigenvector associated with the smallest eigenvalue of $A^{-1}, 1/\lambda_1$, and choosing δb to be an eigenvector associated with the largest eigenvalue of $A^{-1}, 1/\lambda_2$. In particular, $b = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and $\delta b = \frac{\epsilon}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then $\|x\| = \left\|\frac{1}{\lambda_1} b\right\| = \frac{1}{3}$, and $\|\delta x\| = \left\|\frac{1}{\lambda_2}\delta b\right\| = \epsilon$. Thus $\|\delta x\| / \|x\| = 3\epsilon$.

5. Let A and B be $n \times n$ complex matrices with both having characteristic polynomial equal to $f(x) = (x - c_1)^{d_1} \cdots (x - c_k)^{d_k}$, where $1 \leq d_i \leq 3$ for $1 \leq i \leq k$. Prove that A and B are similar if and only if they have the same minimal polynomial. Give two 4×4 matrices A and B which are not similar but which have the same minimal and characteristic polynomials.

Solution: We know that A and B are similar if and only if they have the same Jordan form. Moreover, the Jordan form is the direct sum of the Jordan forms for each of the eigenvalues $c_1, \ldots, c_k$. If J is the Jordan form for a complex matrix with precisely one eigenvalue c, where c has algebraic multiplicity d at most three, then J is completely determined as soon as its minimal polynomial is fixed. Then the characteristic poly of J is $(x-c)^d$, $1 \leq d \leq 3$, and the minimal polynomial is $(x-c)^r$, where $1 \leq r \leq d$.

Case 1. The minimal polynomial of J is $p(x) = (x-c)$. In this case J is the direct sum of d copies of $[c]$, so there are three possibilities.

Case 2. The minimal polynomial of J is $p(x) = (x-c)^2$. In this case J is the direct sum of $\begin{bmatrix} c & 1 \\ 0 & c \end{bmatrix}$ and zero or one copy of $[c]$ depending on whether $d = 2$ or $d = 3$, so there are two possibilities.

Case 3. The minimal polynomial of J is $p(x) = (x-c)^3$. In this case d is forced to equal 3 and

$$J = \begin{bmatrix} c & 1 & 0 \\ 0 & c & 1 \\ 0 & 0 & c \end{bmatrix}.$$

For the second part of the problem, let

$$A = \begin{bmatrix} c & 1 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & 1 \\ 0 & 0 & 0 & c \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} c & 1 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{bmatrix}.$$

Both of these matrices are in Jordan form and have the same minimal polynomial $(x-c)^2$. But since the Jordan forms are different, A and B are not similar.

6. Let $\mathcal{P}_2(\Re)$ denote the usual vector space of all polynomials with coefficients from the reals and with degree at most 2. Define the real inner product $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$. Find the unique polynomial $q \in \mathcal{P}_2(\Re)$ such that for all $p \in \mathcal{P}_2(\Re)$, $\langle p, q \rangle = \int_0^1 p(x)\cos(\pi x)dx$. (Hint: Applying the Gram-Schmidt process to the basis $\{1, x, x^2\}$ yields the orthonormal basis

$$e_0 = 1;\ e_1 = \sqrt{3}(-1+2x);\ e_2 = \sqrt{5}(1 - 6x + 6x^2).)$$

Solution: Given an orthonormal basis $\{e_0, e_1, e_2\}$ for $\mathcal{P}_2(\Re)$, the desired vector q can be written in the form

$$q = \langle q, e_0 \rangle e_0 + \langle q, e_1 \rangle e_1 + \langle q, e_2 \rangle e_2 = \phi(e_0)e_0 + \phi(e_1)e_1 + \phi(e_2)e_2,$$

where ϕ is the functional on $\mathcal{P}_2(\mathcal{R})$ defined by

$$\phi(p) = \int_0^1 p(x)(cos(\pi x))dx.$$

Using the orthonormal basis defined in the hint and evaluating, we have

$$\phi(e_0) = 0, \quad \phi(e_1) = \frac{-4\sqrt{3}}{\pi^2}, \quad \phi(e_2) = 0,$$

so $q(x) = \dfrac{-4\sqrt{3}}{\pi^2} e_1 = \dfrac{12 - 24x}{\pi^2}$.

7. (Vandermonde Determinant)

   Let A be the $n \times n$ matrix whose entries are

$$A = \begin{bmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{n-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & t_n & t_n^2 & \cdots & t_n^{n-1} \end{bmatrix},$$

   where $t_1, \ldots, t_n$ are complex variables.

   Prove that

$$\det A = \prod_{1 \leq j < i \leq n} (t_i - t_j).$$

   (Hint: perform an elementary row operation on A to show that $(t_i - t_j)$ divides $\det A$.)

   Solution: Subtracting row j from row i does not change the determinant and makes it clear that $(t_i - t_j)$ divides the determinant. Hence $\prod_{1 \leq j < i \leq n}(t_i - t_j)$ divides $\det A$. Clearly the determinant is a polynomial in the t_i's having degree $0 + 1 + 2 + \cdots + n - 1 = (n-1)n/2$. This is also the degree of the product, so to show equality we need only check the coefficient on a specific term, say the term $1 \cdot t_2 \cdot t_3^2 \cdots t_n^{n-1}$. Using a standard definition of the determinant,

$$\det A = \sum_{\sigma} (\text{sgn } \sigma) A_{1(\sigma 1)} A_{2(\sigma 2)} \cdots A_{n(\sigma n)}, \tag{1}$$

   the sum being extended over the distinct permutations σ of degree n, we see that the coefficient on this term in $\det A$ is $+1$. We see that the only way to get this term in the product $\prod_{1 \leq j < i \leq n}(t_i - t_j)$ is to choose t_n from all factors that have it, then choose t_{n-1} from all the remaining factors that have it, etc. The coefficient here is then clearly 1.

8. Let A be and $m \times n$ matrix and B be an $n \times p$ matrix. Prove that $\dim \text{null}\,(AB) \leq \dim \text{null}\,(A) + \dim \text{null}\,(B)$. (Hint: In this exercise it is convenient to let $V = \{x \in \mathcal{R}^p : ABx = 0\}$ and $W = \{y = Bx \in \mathcal{R}^n : Ay = 0\}$. Then consider the map $T_B : V \to W$ defined by $T_B : x \mapsto Bx$ for all $x \in V$.)

Solution: Note that AB is an $m \times p$ matrix.

Put $V = \text{null}\,(AB) = \{x \in \mathcal{R}^p : ABx = 0\}$, so $\dim V = \dim \text{null}\,(AB)$.

Put $W = \{Bx : ABx = 0\} \subseteq \text{null}\,(A)$.

Define the linear transformation T_B by:

$$T_B : V \to W : x \mapsto Bx.$$

Clearly the kernel of T_B is a subspace of $\text{null}\,(B)$, so $\dim\ \ker(T_B) \leq \dim \text{null}\,(B)$. (Probably you will notice that the kernel of T_B is exactly $\text{null}\,(B)$, but we don't need this.)

Also, since the image of T_B is contained in W, clearly $\dim \text{Im}(T_B) \leq \dim\,(W) \leq \dim \text{null}\,(A)$. Then since for any linear transformation T, the dimension of its domain equals the dimension of its kernel plus the dimension of its image, we have the following:

$\dim \text{null}\,(AB) = \dim V = \dim\ \ker(T_B) + \dim \text{Im}(T_B) \leq \dim \text{null}\,(B) + \dim \text{null}\,(A)$.