

University of Colorado at Denver — Mathematics Department

Applied Linear Algebra Preliminary Exam

January 13, 2006

Name: \_\_\_\_\_

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value unless otherwise specified.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write legibly using a dark pencil or pen.
- Notation:  $\mathfrak{R}$  denotes the set of real numbers;  $\mathcal{C}$  denotes the set of complex numbers;  $\mathbb{Z}$  denotes the set of integers; and,  $\mathbb{Q}$  denotes the set of rational numbers. These extend to vector spaces as  $\mathfrak{R}^n$ ,  $\mathcal{C}^n$ ,  $\mathbb{Z}^n$ , and  $\mathbb{Q}^n$ , respectively.  $\mathcal{L}(V)$  denotes the set of linear operators on  $V$ . Other notation will be defined as needed.
- Ask the proctor if you have any questions.

Good luck!

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| 2. _____ | 6. _____ |
| 3. _____ | 7. _____ |
| 4. _____ | 8. _____ |

Total \_\_\_\_\_

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

1. Let

$$A = \begin{bmatrix} -10 & 36 & 36 \\ -3 & 11 & 9 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 9 & 5 & -6 \\ 9 & 3 & -4 \end{bmatrix}.$$

Are  $A$  and  $B$  similar matrices? Why or why not?

2. Let  $V$  be a complex inner product space. For  $T \in \mathcal{L}(V)$ , define  $H = \frac{1}{2}(T + T^*)$  and  $S = \frac{1}{2}(T - T^*)$ . Prove that  $T$  is normal if every eigenvector of  $H$  is also an eigenvector of  $S$ .

3. Let  $A$  be an  $m \times n$  real matrix and  $b \in \mathfrak{R}^m$ . Show that exactly one of the following systems has a solution:

- i)  $Ax = b$
- ii)  $A^T y = 0, \quad y^T b \neq 0$ .

Note: Our notation is  $y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$ , so  $y^T = [y_1, \dots, y_m]$ .

4. Consider the matrix  $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ . Let  $b \in \mathfrak{R}^2$  be a unit vector and let  $\delta b \in \mathfrak{R}^2$  be a small perturbation vector, with  $\|\delta b\| = \epsilon > 0$ . Let  $x$  solve  $Ax = b$ , and let  $\delta x$  be the error associated with the perturbation  $\delta b$ , which satisfies  $A(x + \delta x) = b + \delta b$ .

- (a) Calculate the condition number of  $A$ .
- (b) Determine the least upper bound on the relative error  $\frac{\|\delta x\|}{\|x\|}$ .
- (c) Find a  $b$  and  $\delta b$  for which this upper bound is achieved (subject to  $\|b\| = 1$  and  $\|\delta b\| = \epsilon$ ).

5. Let  $A$  and  $B$  be  $n \times n$  complex matrices with both having characteristic polynomial equal to  $f(x) = (x - c_1)^{d_1} \cdots (x - c_k)^{d_k}$ , where  $1 \leq d_i \leq 3$  for  $1 \leq i \leq k$ . Prove that  $A$  and  $B$  are similar if and only if they have the same minimal polynomial. Give two  $4 \times 4$  matrices  $A$  and  $B$  which are not similar but which have the same minimal and characteristic polynomials.

6. Let  $\mathcal{P}_2(\mathfrak{R})$  denote the usual vector space of all polynomials with coefficients from the reals and with degree at most 2. Define the real inner product  $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$ . Find the unique polynomial  $q \in \mathcal{P}_2(\mathfrak{R})$  such that for all  $p \in \mathcal{P}_2(\mathfrak{R})$ ,  $\langle p, q \rangle = \int_0^1 p(x) \cos(\pi x) dx$ . (Hint: Applying the Gram-Schmidt process to the basis  $\{1, x, x^2\}$  yields the orthonormal basis

$$e_0 = 1; \quad e_1 = \sqrt{3}(-1 + 2x); \quad e_2 = \sqrt{5}(1 - 6x + 6x^2).$$

## 7. (Vandermonde Determinant)

Let  $A$  be the  $n \times n$  matrix whose entries are

$$A = \begin{bmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{n-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & t_n & t_n^2 & \cdots & t_n^{n-1} \end{bmatrix},$$

where  $t_1, \dots, t_n$  are complex variables.

Prove that

$$\det A = \prod_{1 \leq j < i \leq n} (t_i - t_j).$$

(Hint: perform an elementary row operation on  $A$  to show that  $(t_i - t_j)$  divides  $\det A$ .)

8. Let  $A$  be an  $m \times n$  matrix and  $B$  be an  $n \times p$  matrix. Prove that  $\dim \text{null}(AB) \leq \dim \text{null}(A) + \dim \text{null}(B)$ . (Hint: In this exercise it is convenient to let  $V = \{x \in \mathcal{R}^p : ABx = 0\}$  and  $W = \{y = Bx \in \mathcal{R}^n : Ay = 0\}$ . Then consider the map  $T_B : V \rightarrow W$  defined by  $T_B : x \mapsto Bx$  for all  $x \in V$ .)