University of Colorado at Denver — Mathematics Department

Applied Linear Algebra Preliminary Exam

September 3, 2005

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value unless otherwise specified.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write legibly using a dark pencil or pen.
- Notation: $\Re$ denotes the set of real numbers; $\mathcal{C}$ denotes the set of complex numbers; $\mathbb{Z}$ denotes the set of integers; and, $\mathbb{Q}$ denotes the set of rational numbers. These extend to vector spaces as $\Re^n$, $\mathcal{C}^n$, $\mathbb{Z}^n$, and $\mathbb{Q}^n$, respectively. Other notation will be defined as needed.
- Ask the proctor if you have any questions.

Good luck!

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________	8.	________
		Total	________

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1. Let $\mathcal{P}_2([0,1])$ be the space of all polynomials of degree 2 or less on the interval $[0,1]$, with inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle p, q \rangle = \int_0^1 p(x)q(x)dx.$$

Let U be the subspace of $\mathcal{P}_2([0,1])$ spanned by $\{x^2, x\}$. Apply the Gram-Schmidt procedure to the basis $\{x^2, x\}$ to produce an orthonormal basis of U.

Solution Observe that $\|x^2\| = \sqrt{\langle x^2, x^2 \rangle} = \sqrt{\int_0^1 x^4 dx} = 1/\sqrt{5}$, so the first element of the orthonormal basis is $u_1 = \sqrt{5}x^2$. The second element of the orthonormal basis is given by $u_2 = w_2 / \|w_2\|$, where

$$\begin{aligned} w_2 &= x - \langle x, u_1 \rangle u_1 \\ &= x - \left(\int_0^1 \sqrt{5}x^3 dx \right) \sqrt{5}x^2 = x - \frac{5}{4}x^2. \end{aligned}$$

$\|w_2\| = \sqrt{\int_0^1 (x - \frac{5}{4}x^2)^2 dx} = \frac{1}{\sqrt{48}}$, so $u_2 = \frac{1}{\sqrt{48}}(x - \frac{5}{4}x^2)$.

In summary, the orthonormal basis is $\left\{ \sqrt{5}x^2, \frac{1}{\sqrt{48}}(x - \frac{5}{4}x^2) \right\}$.

2. Let A and B be two $n \times n$ real matrices satisfying

$$AB = -BA.$$

(a) Prove that A and B cannot both be invertible if n is odd.

(b) Show that A and B can be invertible if n is even.

Solution

(a) $\det(AB) = \det(-BA) = (-1)^n \det(BA) = (-1)^n \det(AB)$. If n is odd, then $\det(AB) = 0$, so one of A or B must not be invertible.

(b) Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Then, $AB = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and $BA = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = -AB$.

3. Let A be a real, symmetric $n \times n$ matrix which is not just a scalar multiple of the identity matrix. Let $f(x) = (x-2)^2(x+5)^3$ and suppose that $f(A) = 0$ and the trace of A is 0.

(a) Determine the minimal polynomial of A.

(b) Determine the characteristic polynomial of A.

(c) Determine the trace of A^2.

(d) Show that n is a multiple of 7.

Solution Since A is real symmetric, its minimal polynomial has no repeated factors, and since $f(A) = 0$ the minimal polynomial divides $f(x)$. Since A is not a scalar times the identity, the minimal polynomial of A has to be exactly $f(x) = x^2 + 3x - 10$. This implies that $A^2 = -3A + 10I$. So the trace of A^2 is $-3(\text{trace}(A)) + 10n = 10n$. As eigenvalues of A, suppose 2 has multiplicity u and -5 has multiplicity v. So $u + v = n$ and $2u - 5v = \text{trace}(A) = 0$.

Solving this system of two linear equations in the two unknowns u and v we find $u = \frac{5n}{7}$ and $v = \frac{2n}{7}$, both of which are positive integers. So there is some positive integer k for which $n = 7k$, $u = 5k$, $v = 2k$. So the characteristic polynomial is

$$c_A(x) = (x-2)^{5k}(x+5)^{2k}.$$

4. Let V and W be finite dimensional vector spaces with inner products $\langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_W$, respectively. Let $T : V \to W$ be a linear transformation with adjoint T^*. Let V_0 be a basis for the null space of T, and V_I be a basis for the image of T^*. Prove that $V_0 \bigcup V_I$ is a basis for V.

Solution: Observe that for any $v \in \text{Null } T$ and $T^*w \in \text{Im } T^*$,

$$\langle v, T^*w \rangle_V = \langle Tv, w \rangle_W = \langle 0, w \rangle_W = 0. \quad (*)$$

Thus, Null $T \perp$ Im T^*.

First, we show that $V_0 \bigcup V_I$ is linearly independent. Let $V_0 = \{v_1, \ldots, v_p\}$ and $V_I = \{T^*w_1, \ldots, T^*w_k\}$, and suppose that scalars $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots \beta_k$ satisfy

$$\alpha_1 v_1 + \cdots \alpha_p v_p + \beta_1 T^* w_1 + \cdots \beta_k T^* w_k = 0.$$

Define $v = \sum_{i=1}^p \alpha_i v_i$. Then

$$\begin{aligned} & 0 = v + \sum_{j=1}^k \beta_j T^* w_j \\ \Rightarrow \quad & \langle v, 0 \rangle_V = \langle v, v \rangle_V + \sum_{b=1}^k \beta_k \langle v, T^* w_k \rangle_V \\ \Rightarrow \quad & 0 = \langle v, v \rangle_V + 0 \quad \text{by } (*) \\ \Rightarrow \quad & v = 0. \end{aligned}$$

Thus, $v = \sum_{i=1}^p \alpha_i v_i = 0$ and $\sum_{j=1}^k \beta_j T^* w_k = 0$. By the linear independence of V_0 and V_I, $\alpha_i = 0$ and $\beta_j = 0$ for all i, j. This establishes the linear independence of $V_0 \bigcup V_I$.

Finally, we show that $V_0 \bigcup V_I$ spans V. For any $v \in V$, let x be the orthogonal projection of v onto Im T^*. Then $y = v - x$ is orthogonal to Im T^*. It is sufficient to show that $y \in \text{Null } T$. Since y is orthogonal to $\text{Im}T^*$, then $\langle y, T^*w\rangle_V = 0$ for any $w \in W$. Choosing $w = Ty$, we have $\langle y, T^*(Ty)\rangle_V = 0 \Rightarrow \langle Ty, Ty\rangle_V = 0 \Rightarrow Ty = 0$. Thus, $y \in \text{Null } T$.

5. Let M be a real symmetric positive semi-definite 4×4 matrix with eigenvalues in the interval $[a, b]$, and let A be the 2×2 matrix that is the top left block of M. Prove that A has eigenvalues in the interval $[a, b]$.

   Solution: A is symmetric (trivially) so has real eigenvalues c and d with $c \leq d$. Let $x \in \Re^2$ be an eigenvector associated with d, and define $w = (x_1, x_2, 0, 0)^T$. Then $b\|w\|_2 \geq \|Mw\|_2 \geq \|Ax\|_2 = d\|x\|_2 = d\|w\|_2$. It follows that $d \leq b$. A similar argument establishes that $a \leq c$.

6. Let A be a square matrix. Show that A is diagonalizable if and only if there is a positive definite Hermitian matrix P such that $P^{-1}AP$ is normal. (Hint: If $A = S\Lambda S^{-1}$, apply the polar decomposition to S).

   Solution: If A is diagonalizable, then we can write $A = S\Lambda S^{-1}$, where Λ is diagonal. Let $S = PU$ be the polar decomposition of S, where P is positive semidefinite and U is unitary. Since S is invertible, P is also invertible, and hence positive definite. Thus, $A = PU\Lambda U^{-1}P^{-1}$, and $P^{-1}AP = U\Lambda U^{-1}$, which is normal.

   Conversely, if there is a positive definite P such that $P^{-1}AP$ is normal, then $P^{-1}AP$ is unitarily diagonalizable. I.e., there exists unitary matrix U and diagonal matrix Λ such that $P^{-1}AP = U\Lambda U^{-1}$. It follows that $A = PU\Lambda U^{-1}P^{-1}$. I.e., A is diagonalizable.

7. Suppose A is a normal matrix such that $A^7 = A^6$. Prove that A is Hermitian and is a projection matrix.

   Solution Suppose A is $n \times n$. Since A is normal, it has an orthogonal set of n eigenvectors $\{v_1, \ldots, v_n\}$. Let λ_i be the eigenvalue associated with v_i. Then $A^7 v_i = \lambda_i^7 v_i = \lambda_i^6 v_i (= A^6 v_i)$. Thus, $\lambda_i = 0$ or $\lambda_i = 1$. Since A is normal with real eigenvalues, it is Hermitian. Further, $\lambda_i^2 = \lambda_i$ (since λ_i is 0 or 1), so $A^2 v_i = Av_i$ and $v_i \in \text{Null}(A^2 - A)$, for all i. Since $\{v_1, \ldots, v_n\}$ is a basis for $\mathcal{C}^n$, it follows that $\text{Null}(A^2 - A) = \mathcal{C}^n$, so $A^2 - A = 0$, and $A^2 = A$; so A is a projection matrix.

8. Prove that the set of 2-by-2 real symmetic matrices is a closed subset of all 2-by-2 real matrices with respect to the spectral norm.

Solution: Let A^n be a sequence of 2-by-2 real symmetic matrices that converges to matrix A with respect to the spectral norm. The set of all 2-by-2 real matrices is a finite dimesional vector space, so all norms are equivalent, e.g., a convergence in the spectral norm is equivalent to the entry-wise convergence. Thus, $A_{ij}^n \to A_{ij}$ and, since $A_{ij}^n = A_{ji}^n$, we have $A_{ij} = A_{ji}$, i.e. A is symmetric.