University of Colorado at Denver — Mathematics Department

Applied Linear Algebra Preliminary Exam

September 3, 2005

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value unless otherwise specified.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write legibly using a dark pencil or pen.
- Notation: $\Re$ denotes the set of real numbers; $\mathcal{C}$ denotes the set of complex numbers; $\mathbb{Z}$ denotes the set of integers; and, $\mathbb{Q}$ denotes the set of rational numbers. These extend to vector spaces as $\Re^n$, $\mathcal{C}^n$, $\mathbb{Z}^n$, and $\mathbb{Q}^n$, respectively. Other notation will be defined as needed.
- Ask the proctor if you have any questions.

Good luck!

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________	8.	________
		Total	________

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1. Let $\mathcal{P}_2([0,1])$ be the space of all polynomials of degree 2 or less on the interval $[0,1]$, with inner product $\langle\cdot,\cdot\rangle$ defined by

$$\langle p, q\rangle = \int_0^1 p(x)q(x)dx.$$

   Let U be the subspace of $\mathcal{P}_2([0,1])$ spanned by $\{x^2, x\}$. Apply the Gram-Schmidt procedure to the basis $\{x^2, x\}$ to produce an orthonormal basis of U.

2. Let A and B be two $n \times n$ real matrices satisfying

$$AB = -BA.$$

   (a) Prove that A and B cannot both be invertible if n is odd.

   (b) Show that A and B can be invertible if n is even.

3. Let A be a real, symmetric $n \times n$ matrix which is not just a scalar multiple of the identity matrix. Let $f(x) = (x-2)^2(x+5)^3$ and suppose that $f(A) = 0$ and the trace of A is 0.

   (a) Determine the minimal polynomial of A.

   (b) Determine the characteristic polynomial of A.

   (c) Determine the trace of A^2.

   (d) Show that n is a multiple of 7.

4. Let V and W be finite dimensional vector spaces with inner products $\langle\cdot,\cdot\rangle_V$ and $\langle\cdot,\cdot\rangle_W$, respectively. Let $T : V \to W$ be a linear transformation with adjoint T^*. Let V_0 be a basis for the null space of T, and V_I be a basis for the image of T^*. Prove that $V_0 \bigcup V_I$ is a basis for V.

5. Let M be a real symmetric positive semi-definite 4×4 matrix with eigenvalues in the interval $[a,b]$, and let A be the 2×2 matrix that is the top left block of M. Prove that A has eigenvalues in the interval $[a,b]$.

6. Let A be a square matrix. Show that A is diagonalizable if and only if there is a positive definite Hermitian matrix P such that $P^{-1}AP$ is normal. (Hint: If $A = S\Lambda S^{-1}$, apply the polar decomposition to S).

7. Suppose A is a normal matrix such that $A^7 = A^6$. Prove that A is Hermitian and is a projection matrix.

8. Prove that the set of 2-by-2 real symmetic matrices is a closed subset of all 2-by-2 real matrices with respect to the spectral norm.