Analysis Preliminary Exam Committee:

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1. Prove that if $f(x)$ and $g(x)$ are continuous then $h(x) = f(x)g(x)$ is also continuous using the $\epsilon - \delta$ (epsilon-delta) definition of continuity. (No points for not using the epsilon-delta definition of continuity - do **not** use the sequential definition).

   Solution

   Choose $\epsilon > 0$. For a given x we will show that $\exists \delta \in (0,1)$ such that $|x - y| < \delta$ implies $|h(x) - h(y)| < \epsilon$. Let $K > \max_{y \in [x-1, x+1]} \{f(y), g(y)\}$. Then if $|x - y| < 1$ we have

$$\begin{aligned} |h(x) - h(y)| &= |f(x)g(x) - f(y)g(y)| \\ &= |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \\ &\leq |f(x)||g(x) - g(y)| + |g(y)|f(x) - f(y)| \end{aligned}$$

   Choose $\delta \in (0,1)$ so that $|x - y| < \delta$ inplies $|g(x) - g(y)| < \epsilon/2K$ and $|f(x) - f(y)| < \epsilon/2K$ which we can do since f and g are continuous. The result follows.

2. Prove that if a real-valued function $f : D \subseteq \Re \to \Re$ is continuous, then the image of a compact set is compact.

Solution

Let $K \subset D$ be compact. We must show that $f(K)$ is compact. There are many ways to prove this result, depending on which theorems are taken as givens. The best proof is purely topological:

Let $\{A_\alpha\}$ be an open cover for $f(K)$. Then $\{f^{-1}(A_\alpha)\}$ is an open cover for K. Since K is compact, we can find $\{f^{-1}(A_{\alpha_1}), \ldots, f^{-1}(A_{\alpha_n})\}$ that cover K, so $\{A_{\alpha_1}, \ldots, A_{\alpha_n}\}$ cover $f(K)$, implying that $f(K)$ is compact.

Here is an $\epsilon - \delta$ proof based on Heine-Borel: We must show that $f(K)$ is closed and bounded.

$f(K)$ is CLOSED: Let $\{y_n\}$ be a sequence in $f(K)$ with $y_n \to y$. We must show that $y \in f(K)$. Let $\{x_n\} \in K$ satisfy $f(x_n) = y_n$. Since K is compact, $\{x_n\}$ has a limit point, $x \in K$. So there is a subsequence $x_{n_i} \to x$. Since f is continuous, $f(x_{n_i}) \to f(x)$. Thus, $y_{n_i} \to f(x)$. But $y_{n_i} \to y$, so $f(x) = y$.

$f(K)$ is BOUNDED: If $f(K)$ is unbounded then $\exists y_n \in f(K)$ with $|y_n| > n$. Let $x_n \in K$ satsify $f(x_n) = y_n$. Since K is compact, $\{x_n\}$ has a limit point $x \in K$ and a subsequence $x_{n_i} \to x$. Since f is continuous, $f(x_{n_i}) \to f(x)$. But $|f(n_{n_i})| \to \infty$, so the original seuqence $\{y_n\}$ must not exist.

3. Prove that if $f : \Re \to \Re$ is continuous and one-to-one then f is strictly monotone.

Solution

If $f(x)$ is not strictly monotone then $\exists x, y, z$ such that $x < y < z$ and either $f(x) \leq f(y) \geq f(z)$ or $f(x) \geq f(y) \leq f(z)$. If $f(x) = f(y)$ or $f(y) = f(z)$ then f is not one-to-one, so we must have $f(x) < f(y) > f(z)$ or $f(x) > f(y) < f(z)$. Suppose $f(x) < f(y) > f(z)$. (The other case is the same.) Let $\delta < \min\{f(y) - f(x), f(y) - f(z)\}$. Since f is continuous, by the intermediate value theorem $\exists a \in (x, y)$ such that $f(a) = f(y) - \delta$ and $\exists b \in (y, z)$ such that $f(b) = f(y) - \delta$, which again implies f is not one-to-one. We conclude that f is strictly monotone.

4. Suppose $f(x)$ is continuous at $c \in (a, b)$ and differentiable on the rest of (a, b). Prove that if $\lim_{x \to c} f'(x)$ exists then $f(x)$ is differentiable at c.

Solution

Need to show

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

converges. Since $f(x)$ is differentiable for $x \neq c$ and since the numerator and denominator go to zero (since f is continuous) we can use L'Hostital's rule:

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{f'(x)}{1} = f'(c)$$

5. Let $B_r(\mathbf{x}_0)$ be a ball of radius r centered at $\mathbf{x}_0$ in the space $\Re^n$. Suppose $f : E \to \Re^n$ is a function which is Riemann integrable on the open set E and is continuous at $\mathbf{x}_0 \in E$. Prove

$$\lim_{r \to 0^+} \frac{1}{|B_r(\mathbf{x}_0)|} \int_{B_r(\mathbf{x}_0)} f(\mathbf{x})\, d\mathbf{x} = f(\mathbf{x}_0),$$

where $|B_r(\mathbf{x}_0)|$ denotes the volume of $|B_r(\mathbf{x}_0)|$.

Solution

Choose $\varepsilon > 0$. Since f is continuous at x_0, $\exists \delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$. Since f is Riemann integrable, for any $r < \delta$,

$$\begin{aligned}
\left| \frac{1}{|B_r(\mathbf{x}_0)|} \int_{B_r(\mathbf{x}_0)} f(x) dx - f(x_0) \right| &= \left| \frac{1}{|B_r(\mathbf{x}_0)|} \int_{B_r(\mathbf{x}_0)} \left(f(\mathbf{x}) - f(\mathbf{x}_0) \right)\, d\mathbf{x} \right| \\
&\leq \frac{1}{|B_r(\mathbf{x}_0)|} \int_{B_r(\mathbf{x}_0)} |f(\mathbf{x}) - f(\mathbf{x}_0)|\, d\mathbf{x} \\
&\leq \frac{1}{|B_r(\mathbf{x}_0)|} \int_{B_r(\mathbf{x}_0)} \varepsilon\, d\mathbf{x} \\
&= \frac{1}{|B_r(\mathbf{x}_0)|} \varepsilon\, |B_r(\mathbf{x}_0)| = \varepsilon.
\end{aligned}$$

6. Let $E \subset \Re$ and define $C_b(E)$ to be the space of bounded continuous functions $f : E \longrightarrow \Re$, with the sup norm:

$$||f|| = \sup_{x \in E} |f(x)|.$$

   (a) Prove that $C_b(E)$ is a metric space (i.e., verify the axioms).

   (b) Prove that $C_b(E)$ is a *complete* metric space.

Solution

(From pages 147-151 Rudin)

(a) To show $C_b(E)$ is a metric space we must verify the three properties. Suppose f and g are bounded continuous functions over E:

   i. If $f \neq g$ then there is x such that $(f-g)(x) \neq 0$ which implies

$$0 < |(f-g)(x)| < ||f-g|| = d(f,g).$$

   Also $d(f,f) = \sup_{x \in E} |f - f| = \sup_{x \in E} 0 = 0$.

   ii. Symmetry: $d(f,g) = \sup_{x \in E} |f - g| = \sup_{x \in E} |g - f| = d(g,f)$.

   iii. Triangle inequality: Let $h \in C_b(E)$. Then for each fixed x we have

$$|(f-g)(x)| < |(f-h)(x)| + |(h-g)(x)| \leq ||f-h|| + ||h-g||$$

   where the second inequality is due to the definition of the sup norm. Since the right side is independent of x we can take the sup of the left side and obtain the desired result: $||(f-g)(x)|| \leq ||f-h|| + ||h-g||$.

(b) We must show that every Cauchy sequence converges to a function in $C_b(E)$. First we need a candidate. Let f_n be a Cauchy sequence, i.e. given $\varepsilon > 0$ there exists N such that for all $n, m \geq N$ we have $||f_n - f_m|| < \varepsilon$. Thus for fixed x we have

$$|f_n(x) - f_m(x)| \leq ||f_n - f_m|| < \varepsilon.$$

So for each fixed x we have a Cauchy sequence. Since $\Re$ is a complete metric space we know that $f_n(x)$ converges to a value in $\Re$ and so let $f(x)$ be defined to be the pointwise limit of $f_n(x)$.

Claim: f is bounded: Let $\varepsilon = 1$ then since f_n is a Cauchy sequence, there is N such that for all $n \geq N$ we have $||f_n - f_N|| < 1$, or $\forall n \geq N$ and $\forall x \in E$ we have $-1 < f_n(x) - f_N(x) < 1$. Thus the limit as n goes to infinity is bounded by $|f_N| + 1$ so f is bounded for all $x \in E$.

Claim: f is continuous: Let ε be given. Then there exists N such that $\forall n > N$ we have $||f_N - f_n|| < \varepsilon/3$, or $-\varepsilon/3 < f_n - f_N < \varepsilon/3$. Taking the limit as n goes to ∞, and using the fact that the inequality holds for all $x \in E$, we have $||f - f_N|| < \varepsilon/3$. Now f_N is continous, so given the same ε there exists δ such that if $|x - y| < \delta$ we have $|f_N(x) - f_N(y)| < \varepsilon/3$. Thus for $|x - y| < \varepsilon$ we have

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

7. Evaluate

$$\lim_{n\to\infty} e^{-nx}\left(1+\frac{x}{n}\right)^{n^2}$$

Justify every important step.

Solution

Let $a_n = e^{-nx}(1+x/n)^{n^2}$, so

$$\begin{aligned}
\ln a_n &= -nx + n^2 \ln(1+x/n) \\
&= -nx + n^2(x/n - x^2/2n^2 + x^3/3n^3 \cdots) \\
&= x^2/2 - \sum_{k=3}^{\infty} (-1)^k x^k / kn^{k-2} \\
&= x^2/2 - x^3/n \sum_{k=3}^{\infty} (-1)^k x^{k-3}/kn^{k-3}
\end{aligned}$$

If $n > x$ then $\sum_{k=3}^{\infty}(-1)^k x^{k-3}/kn^{k-3}$ is a decreasing alternating sequence, which converges to something. Thus

$$x^3/n \sum_{k=3}^{\infty}(-1)^k x^{k-3}/kn^{k-3} \to 0,$$

so $|\ln a_n - x^2/2| \to 0$, so $a_n \to e^{x^2/2}$.

8. Suppose we are given a sequence of differentiable functions, $\{f_n\}$, $f_n : \Re \to \Re$, such that the sequence of derivatives $\{f'_n\}$, $f'_n : \Re \to \Re$, is uniformly convergent and the sequence of numbers $\{f_n(0)\}$ is also convergent

   (a) Prove that the sequence $\{f_n\}$ is pointwise convergent.

   (b) Show that the assumption that the sequence $\{f_n(0)\}$ is convergent is necessary by giving a counterexample that demonstrates this.

   Solution

   By assumption we have $f_n(0) \to f(0)$ and $f'_n(y) \to f'(y)$ uniformly.

$$f_n(x) = f_n(0) + \int_0^x f'_n(y)dy$$

so

$$\lim f_n(x) = \lim f_n(0) + \lim \int_0^x f'_n(y)dy.$$

Since f'_n converges uniformly,

$$\lim \int_0^x f'_n(y)dy = \int_0^x \lim f'_n(y)dy = \int_0^x f'(y)dy = f(x) - f(0)$$

and the result follows.

To show that $f_n(0)$ must converge for the result to be true, consider $f_n(x) = n$. Then $f'_n(x) = 0$ which (obviously) converges to 0 uniformly, but $f_n(x) \to \infty$.