University of Colorado at Denver — Mathematics Department

Applied Analysis Preliminary Exam

January 8, 2007

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit **no more than** 6 solutions. If you submit more than 6 solutions only the first six problems (as determined by the numbering of the problems) will be graded.
- Each problem is worth 20 points; parts of problems have equal value.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write legibly using a dark pencil or pen.
- Notation: $\mathbb{R}$ denotes the set of real numbers; $\mathbb{Z}$ denotes the set of integers; and, $\mathbb{C}$ denotes the set of complex numbers. These extend to vector spaces as $\mathbb{R}^n$, $\mathbb{Z}^n$, and $\mathbb{C}^n$, respectively. Other notation will be defined as needed.
- Ask the proctor if you have any questions.

Good luck!

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________	8.	________
		Total	________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Analysis Preliminary Exam Committee:
Lynn Bennethum, Burt Simon, Weldon Lodwick (Chair)

1. Prove that if $f(x)$ and $g(x)$ are continuous then $h(x) = f(x)g(x)$ is also continuous using the $\epsilon - \delta$ (epsilon-delta) definition of continuity. (No points for not using the epsilon-delta definition of continuity - do **not** use the sequential definition).

2. Prove that if a real-valued function $f : D \subseteq \Re \to \Re$ is continuous, then the image of a compact set is compact.

3. Prove that if $f : \Re \rightarrow \Re$ is continuous and one-to-one then f is strictly monotone.

4. Suppose $f(x)$ is continuous at $c \in (a, b)$ and differentiable on the rest of (a, b). Prove that if $\lim_{x \to c} f'(x)$ exists then $f(x)$ is differentiable at c.

5. Let $B_r(\mathbf{x}_0)$ be a ball of radius r centered at $\mathbf{x}_0$ in the space $\Re^n$. Suppose $f : E \to \Re^n$ is a function which is Riemann integrable on the open set E and is continuous at $\mathbf{x}_0 \in E$. Prove

$$\lim_{r \to 0^+} \frac{1}{|B_r(\mathbf{x}_0)|} \int_{B_r(\mathbf{x}_0)} f(\mathbf{x})\, d\mathbf{x} = f(\mathbf{x}_0),$$

where $|B_r(\mathbf{x}_0)|$ denotes the volume of $|B_r(\mathbf{x}_0)|$.

6. Let $E \subset \Re$ and define $C_b(E)$ to be the space of bounded continuous functions $f : E \longrightarrow \Re$, with the sup norm:

$$||f|| = \sup_{x \in E} |f(x)|.$$

   (a) Prove that $C_b(E)$ is a metric space (i.e., verify the axioms).

   (b) Prove that $C_b(E)$ is a *complete* metric space.

7. Evaluate
$$\lim_{n\to\infty} e^{nx}\left(1+\frac{x}{n}\right)^{n^2}$$
Justify every important step.

8. Suppose we are given a sequence of differentiable functions, $\{f_n\}$, $f_n : \Re \to \Re$, such that the sequence of derivatives $\{f'_n\}$, $f'_n : \Re \to \Re$, is uniformly convergent and the sequence of numbers $\{f_n(0)\}$ is also convergent

   (a) Prove that the sequence $\{f_n\}$ is pointwise convergent.

   (b) Show that the assumption that the sequence $\{f_n(0)\}$ is convergent is necessary by giving a counterexample that demonstrates this.