

University of Colorado at Denver — Mathematics Department

Applied Analysis Preliminary Exam

January 22, 2005

Name: \_\_\_\_\_

**Exam Rules:**

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write legibly using a dark pencil or pen.
- Notation:  $\mathbb{R}$  denotes the set of real numbers;  $\mathbb{Z}$  denotes the set of integers; and,  $\mathbb{C}$  denotes the set of complex numbers. These extend to vector spaces as  $\mathbb{R}^n$ ,  $\mathbb{Z}^n$ , and  $\mathbb{C}^n$ , respectively. Other notation will be defined as needed.
- Ask the proctor if you have any questions.

Good luck!

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| 1. _____ | 5. _____ |
| 2. _____ | 6. _____ |
| 3. _____ | 7. _____ |
| 4. _____ | 8. _____ |

Total \_\_\_\_\_

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**Analysis Preliminary Exam Committee:**

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1. Let  $X$  and  $Y$  be metric spaces,  $f : X \rightarrow Y$  continuous, and  $E \subset X$ . Show that

$$f(\overline{E}) \subset \overline{f(E)}$$

where  $\overline{E}$  means the closure of  $E$ . Give an example where  $f(\overline{E}) \neq \overline{f(E)}$ .

**Solution**

Because the closure of  $E$  consists of the points of  $E$  and limit points of  $E$ , the closure is the set of all limits of sequences from  $E$ . So, let  $e \in \overline{E}$ ; then there exists a sequence  $e_n \rightarrow e$  in  $X$ . Because  $f$  is continuous,  $f(e_n) \rightarrow f(e)$  in  $Y$ . Because  $f(e_n) \in f(E)$ , it follows that  $f(e) \in \overline{f(E)}$ . This proves that  $f(\overline{E}) \subset \overline{f(E)}$ . Note that the closure  $\overline{E}$  is in the space  $X$  while the closure  $\overline{f(E)}$  is taken in the space  $Y$ .

To show that the inclusion can be proper, consider  $X = (0, 1)$ ,  $Y = \mathbb{R}$  both with the usual Euclidean distance,  $f(x) = x$ , and  $E = X$ . Then  $E \subset \overline{E} \subset X$  so  $\overline{E} = X = (0, 1)$  and  $f(E) = (0, 1)$ , but in  $Y$ ,  $\overline{f(E)} = \overline{(0, 1)} = [0, 1]$ .

2. Prove that for  $n \geq 2$ ,

$$\ln(n) - \sum_{k=2}^n \frac{1}{k} < 1/2.$$

**Solution**

Since  $1/x$  is convex,

$$\int_{k-1}^k \frac{dx}{x} < \frac{1}{2} \left( \frac{1}{k-1} + \frac{1}{k} \right),$$

for  $k > 1$ , so

$$\ln n = \int_1^n \frac{dx}{x} = \sum_{k=2}^n \int_{k-1}^k \frac{dx}{x} < \sum_{k=2}^n \frac{1}{2} \left( \frac{1}{k-1} + \frac{1}{k} \right).$$

Thus,

$$\begin{aligned} \ln n - \sum_{k=2}^n \frac{1}{k} &< \sum_{k=2}^n \frac{1}{2} \left( \frac{1}{k-1} + \frac{1}{k} \right) - \sum_{k=2}^n \frac{1}{k} \\ &= \frac{1}{2} \sum_{k=2}^n \left( \frac{1}{k-1} - \frac{1}{k} \right) = \frac{1}{2} \left( 1 - \frac{1}{n} \right) \\ &< \frac{1}{2}. \end{aligned}$$

3. Let  $f$  be differentiable on  $[a, b]$  with  $|f'(x)| < \beta$ . Let  $P$  be a partition of  $[a, b]$ , and let  $U(f, P)$  and  $L(f, P)$  be the upper and lower Riemann sums. Prove that

$$U(f, P) - L(f, P) \leq \beta|P|(b - a)$$

**Solution**

Since  $|f'| < \beta$ , the mean value theorem yields

$$|f(x) - f(y)| \leq \beta|x - y|$$

Let  $\alpha_i$  and  $\beta_i$  be the position of a (resp.) maximum and minimum of  $f(x)$  on the  $i$ th interval of partition  $P$ . Thus,

$$f(\alpha_i) - f(\beta_i) \leq \beta|\alpha_i - \beta_i| \leq \beta|x_i - x_{i-1}| \leq \beta|P|$$

Thus,

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^n (f(\alpha_i) - f(\beta_i))(x_i - x_{i-1}) \\ &\leq \beta|P| \sum_{i=1}^n (x_i - x_{i-1}) = \beta|P|(b - a). \end{aligned}$$

4. Construct a function  $f$  on  $[0, \infty]$  satisfying

- (a)  $f$  continuous
- (b)  $0 < f < 1$
- (c)  $\int_0^\infty f(x)dx < 1$
- (d)  $\limsup_{x \rightarrow \infty} f(x) = 1$

Note: The items in the above list are *not* parts of the problem with equal value as mentioned on the front page. This is a single problem to construct a function that satisfies *all* of the above conditions.

### Solution

We will construct a function that has a series of bumps of size approaching 1, while the bumps are getting narrower to control the value of the integral, and take maximum with a positive function with a small integral to guarantee that the result is positive. Let

$$g_n(x) = \begin{cases} 0 & \text{if } x \notin (n - 1/2^n, n + 1/2^n) \\ (1 - 1/n)(1 - 2^n|x - n|) & \text{if } x \in (n - 1/2^n, n + 1/2^n) \end{cases}$$

Let

$$f(x) = \max\{1/10e^{-x}, g_2(x), g_3(x), \dots\}$$

Then  $f$  is continuous and positive, and less than 1. Also,

$$\int_0^\infty f(x)dx \leq \frac{1}{10} \int_0^\infty e^{-x} dx + \sum_{n=1}^\infty \frac{1}{2} (1 - 1/n)(1/2^{n-1}) < \frac{1}{10} + \frac{1}{2} < 1$$

Finally, for any  $t > 0$ ,  $\sup_{x > t} f(x) = 1$ , so  $\limsup_{x \rightarrow \infty} f(x) = 1$ .

5. Let  $I$  be a closed and bounded interval in  $\mathbb{R}$  and suppose that  $g(x) : I \rightarrow I$  is a continuous function on  $I$  with a continuous first derivative on  $I$  such that  $|g'(x)| < 1 \forall x \in I$ . Show that the sequence defined by

$$x_0 \in I, \quad x_{n+1} = g(x_n)$$

converges to the unique fixed point of  $g$ , i.e.  $\lim_{n \rightarrow \infty} x_n = x^*$  where  $g(x^*) = x^*$ .

**Solution**

Because  $g'$  is continuous on a closed and bounded interval  $I$  (hence, a compact set),  $g'$  attains its maximum,  $\lambda = g'(x_0) < 1$ . Then, from the mean value theorem, for any  $x, y \in I$ ,

$$g(y) - g(x) = g'(c)(y - x)$$

where  $c \in [x, y]$ . This gives us

$$|g(y) - g(x)| \leq \lambda|y - x|, \forall x, y \in I.$$

Now use the Banach Fixed Point theorem: *If  $g : I \rightarrow I$  where  $I$  is a complete metric space with distance function  $d$  and  $g$  is a contraction,*

$$d(g(y), g(x)) \leq \lambda d(y, x), \forall x, y \in I$$

for some  $\lambda < 1$ , then the sequence defined by

$$x_0 \in I, \quad x_{n+1} = g(x_n)$$

converges to the unique fixed point  $x^*$  of  $g$  in  $I$ ,  $\lim_{n \rightarrow \infty} d(x_n, x^*) = 0$ . The metric in  $\mathbb{R}$  is defined by

$$d(x, y) = |x - y|,$$

so  $g$  is a contraction. Since  $I$  is closed and bounded, it is complete, which concludes the proof.

Remarks. One cannot use the fixed point theorem directly when it is only known that  $|g(y) - g(x)| < |y - x|$ . It is important to point out where it is used that  $I$  is closed and where it is used that  $I$  is bounded.

6. Let  $F(x) = f_1(x) + f_2(x) + f_3(x) + \dots$  be a series that is uniformly convergent over  $(a, b)$  and for which each  $f_k(x)$  is continuous on  $[a, b]$ . Assume that we already know that  $f_k(b)$  converges to a value, which we will define to be  $F(b)$ . Prove that

$$F(b) = \lim_{x \rightarrow b^-} F(x).$$

### Solution

We need to show that given  $\epsilon > 0$ , there exists  $\delta > 0$  such that for all  $b - \delta < x < b$ , it holds that  $|F(b) - F(x)| < \epsilon$ . Let  $F_k(x) = f_1(x) + f_2(x) + f_3(x) + \dots + f_k(x)$ . By the triangle inequality,

$$|F(b) - F(x)| \leq |F(b) - F_k(b)| + |F_k(b) - F_k(x)| + |F_k(x) - F(x)|.$$

The idea of the rest of the proof is: Since we know that  $F_k(b)$  converges (by the definition of  $F(b)$ ), we have a bound on the first term on the right-hand-side. The second term can be bounded by using continuity of  $f_k(x)$  and the third term by using uniform continuity.

So suppose  $\epsilon > 0$  is given. Since  $F_k(b)$  converges to  $F(b)$ , there exists  $N_1(\epsilon)$  such that  $\forall k > N_1, |F_k(b) - F(b)| < \epsilon/3$ . Since  $F_k(x)$  converges uniformly to  $F(x)$  for  $x \in (a, b)$ , there is an  $N_2(\epsilon)$ , independent of  $x$ , so that if  $k > N_2, |F_k(x) - F(x)| < \epsilon/3$ . Let  $N = \max\{N_1, N_2\}$ , so that we have the first and third terms bounded by  $\epsilon/3$  for all  $k > N$  and all  $x$ . Now fix  $k > N$ . Then  $F_k(x)$  is continuous on  $[a, b]$  (it's a finite sum), so there exists  $\delta$  so that if  $b - \delta < x < b$  we have  $|F_k(b) - F_k(x)| < \epsilon/3$ . And we are done.

7. Prove the *Mean-Value Theorem for Integrals*: Let  $u$  and  $v$  be continuous real-valued functions on an interval  $[a, b]$ , and suppose that  $v \geq 0$  on  $[a, b]$ . Then there exists a point  $\xi$  in  $[a, b]$  such that

$$\int_a^b u(x) v(x) dx = u(\xi) \int_a^b v(x) dx.$$

**Solution**

Let

$$g(t) = u(t) \int_a^b v(x) dx$$

which is continuous since  $u(t)$  is continuous. Since  $u$  is continuous on a closed and bounded interval, it attains its minimum and maximum. So, let  $x^*$  satisfy  $u(x^*) \geq u(x)$ ,  $x \in (a, b)$  and  $x_*$  satisfy  $u(x_*) \leq u(x)$ ,  $x \in (a, b)$ . Then

$$g(x_*) \leq \int_a^b u(x)v(x)dx \leq g(x^*)$$

By the intermediate value theorem, since  $g$  is continuous, there must be  $\xi$  between  $x^*$  and  $x_*$  so that  $g(\xi) = \int_a^b u(x)v(x)dx$ .

Another solution.

First note that if  $v(x)$  is exactly zero then the above inequality reduces to  $0 = 0$ . So assume that  $v(x)$  is not identically zero and since  $v(x)$  is continuous,  $\int_a^b v(x) dx > 0$ . Since  $u(x)$  is continuous over a closed interval, it has a maximum,  $M$ , and a minimum,  $m$ , which implies

$$\begin{aligned} m &\leq u(x) \leq M \\ mv(x) &\leq u(x)v(x) \leq Mv(x) && (v(x) \geq 0) \\ \int_a^b mv(x) dx &\leq \int_a^b u(x)v(x) dx \leq \int_a^b Mv(x) dx \\ m &\leq \frac{\int_a^b u(x)v(x) dx}{\int_a^b v(x) dx} \leq M. \end{aligned}$$

Now since  $u(x)$  is continuous, the Mean Value Theorem tells us that  $u(x)$  takes on every value between  $m$  and  $M$ . In particular there exists  $\xi$  such that

$$u(\xi) = \frac{\int_a^b u(x)v(x) dx}{\int_a^b v(x) dx}.$$

Thus:

$$\int_a^b u(x) v(x) dx = u(\xi) \int_a^b v(x) dx.$$

8. Give an example of a function  $f(x, y)$  such that

$$\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} f(x, y) \neq \lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x, y)$$

**Solution**

Define

$$f(x, y) = \begin{cases} \frac{y}{1+\frac{y}{x}} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Then  $\lim_{x \rightarrow 0} f(x, y) = 1$  for  $y \neq 0$ , so

$$\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x, y) = 1.$$

On the other hand,  $\lim_{y \rightarrow 0} f(x, y) = 0$  for  $x \neq 0$ , so

$$\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} f(x, y) = 0.$$