**University of Colorado at Denver — Mathematics Department**

**Applied Analysis Preliminary Exam**

**June 2, 2006**

Name: ______________________________________________

**Exam Rules**:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write legibly using a dark pencil or pen.
- Notation: $\mathbb{R}$ denotes the set of real numbers; $\mathbb{Z}$ denotes the set of integers; and, $\mathbb{C}$ denotes the set of complex numbers. These extend to vector spaces as $\mathbb{R}^n$, $\mathbb{Z}^n$, and $\mathbb{C}^n$, respectively. Other notation will be defined as needed.
- Ask the proctor if you have any questions.

Good luck!

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**Analysis Preliminary Exam Committee:**
Lynn Bennethum, Bill Briggs, Jan Mandel (Chair)

1. Show that if $A \subset \mathbb{R}$ is open then $A$ has no isolated points. Show that this statement is not true in a general metric space, i.e., give an example of a metric space $(X, \rho)$, where there exists an open set $A \subset X$ which has an isolated point.

   **Solution**

   Suppose that $a$ is an isolated point of $A$. Then there exists an interval $(b, c)$ such that $a \in (b, c)$ and $(b, c) \cap A = \{a\}$. Because $A$ is open, there exists an interval $(d, e)$ such that $a \in (d, e) \subset A$, but the interval $I = (b, c) \cap (d, e) = (\max\{b, c\}, \min\{c, e\})$ satisfies $I \subset (d, e) \subset A$ so $I \cap A = I$ but at the same time $I \cap A \subset (b, c) \cap A = \{a\}$, which is a contradiction.

   [Note that the argument above also covers the case when $A = \emptyset$; in that case, the the premise "suppose that $a$ is an isolated point of $A$" implies that $A$ is nonempty (which is a contradiction, and we could stop there), but there is nothing wrong with letting the argument run its course and arrive at a different contradiction. Or, one can treat the case of $A = \emptyset$ separately and say that in this case, $A$ has no elements and thus no interior points, because an interior point of $A$ is an element of $A$.]

   The statement is not true in a metric space consisting of a finite number of points, which has no limit points, but the whole metric space is always open in itself.

2. Prove that there is no continuous map from the closed interval $[0,1]$ onto the open interval $(0,1)$. Hint: if $f$ is such a function, then there exists $x_n \in f^{-1}((0,1/n))$.

**Solution**

Assume that $f : [0,1] \to (0,1)$ is continuous and surjective. Since it is surjective there exists $x_n \in f^{-1}((0,1/n)) \in [0,1]$. Because $[0,1]$ is compact, $x_n$ has a subsequence $x_{n_k}$, which converges to $x \in [0,1]$. By continuity of $f$ and from $f(x_n) \in (0,1/n)$, we have

$$f(x) = \lim_{k\to\infty} f(x_{n_k}) = 0$$

But $0 \notin (0,1)$, a contradiction.

Another solution: The interval $[0,1]$ (with the metric $d(x,y) = |x-y|$, same as in $\mathbb{R}$) is a compact metric space, so if $f$ is continuous on $[0,1]$ and $(0,1) = f([0,1])$, then $((0,1),d)$ is also a compact metric space. But the open interval $(0,1)$ is not compact, because it is not complete: a sequence convergent to one of the endpoints is Cauchy in $(0,1)$ but it does not have a limit in $(0,1)$.

[ Note that an argument along the lines "because $f$ is continuous, the inverse image of an open set is open" does not give a contradiction, because every metric space is an open set in itself, so the set $[0,1]$ is open in the metric space $([0,1],d)$. ]

3. Let $\{a_k\}$ be a sequence in $\mathbb{R}$. Using the definitions, prove the following in $\mathbb{R}$: $\sup\{a_k : k \in N\}$ does not exist $\iff$ $\{a_k\}$ is not bounded above $\iff$ $\forall n \in \mathbb{N}$ : $\sup\{a_k : k \geq n\}$ does not exist

   **Solution**

   Let $A$ ="$\sup\{a_k : k \in N\}$ does not exist", $B$ ="$\{a_k\}$ is not bounded above", $C$ ="$\forall n \in \mathbb{N} : \sup\{a_k : k \geq n\}$ does not exist".

   $A \implies B$ : The set $\{a_k : k \in N\}$ is not empty so if its supremum does not exist it is not bounded above.

   $B \implies C$ : We prove the contrapositive, $\neg C \implies \neg B$. Suppose for some $n \in \mathbb{N}$, $s = \sup\{a_k : k \geq n\}$ exists. Then $\forall k \in \mathbb{N} : a_k \leq \max\{s, a_1, \ldots, a_{n-1}\}$ so the sequence $\{a_k\}$ is bounded above.

   $C \implies A$ : Set $n = 1$.

4. Let $f_n : \mathbb{R} \to \mathbb{R}$ be differentiable for each $n = 1, 2, ...$ with $|f_n'(x)| \le 1$ for all $n$ and for all $x$. Assume
$$\lim_{n\to\infty} f_n(x) = g(x) \qquad \forall x \in \mathbb{R}.$$
Prove that $g : \mathbb{R} \to \mathbb{R}$ is continuous.

**Solution**

From the mean value theorem,

$$|f(x) - f(y)| = \left|f'(\xi)\right| |x - y| \le |x - y|,$$

with some $\xi$. Thus the set of functions $\{f_n\}$ is equicontinuous. Since $f_n$ converge pointwise, the $\{f_n\}$ is pointwise bounded. Let $I$ be a compact interval. By the Arzelà-Ascoli theorem, there exists a uniformly convergent subsequence $f_{n_k}$. Since $f_n \to g$ and the pointwise limit is unique, $f_{n_k} \rightrightarrows g$. Since each $f_{n_k}$ is continuous on $I$, $g$ is continuous on $I$. If $x$ is an arbitrary point in $\mathbb{R}$, then $x$ is in a compact interval, say $[x-1, x+1]$, and by the previous argument $g$ is continuous on $[x-1, x+1]$ and thus continuous at $x$.

Another solution:

We need to show: Given $\varepsilon > 0$, there is a $\delta(\varepsilon)$ such that if $|x-y| < \delta(\varepsilon)$ then $|g(x) - g(y)| < \varepsilon$. So let $\varepsilon > 0$ be given. Let $\delta = \varepsilon/2$ and let $|x - y| < \delta = \varepsilon/2$. Since $f_n(x) \to g(x)$, there exists $N_x$ such that $\forall n > N_x$ we have $|f_n(x) - g(x)| < \varepsilon/4$. Likewise, since $f_n(y) \to g(y)$, there exists $N_y$ such that $\forall n > N_y$ we have $|f_n(y) - g(y)| < \varepsilon/4$. Then $\forall n > \max(N_x, N_y)$ we have, by using the Mean Value Theorem:

$$\begin{aligned}
|g(x) - g(y)| &\le |g(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - g(y)| \\
&\le \frac{\varepsilon}{4} + |f'(c)||x-y| + \frac{\varepsilon}{4} \\
&\le \frac{\varepsilon}{2} + |x - y| \\
&\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

In the above $x < c < y$.

5. Let $f_n(x) : [0,1] \to \mathbb{R}$ be a sequence of continuous functions. Suppose that $\lim_{n\to\infty} f_n(x) = 0$ for each $x \in [0,1]$ and also that, for some constant $K$, we have

$$\left| \int_0^1 f_n(x)\, dx \right| \leq K < \infty \qquad \forall n.$$

Does it hold that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = 0?$$

Prove or give a counterexample.

**Solution**

This question essentially asks is it ok to have pointwise convergence and uniformly bounded integrals (instead of uniform convergence) to allow the interchange of a limit and integral? To test this, let's try coming up with a sequence of functions which converges pointwise to 0, but not uniformly, but has the same integrals over $[0,1]$. Consider a sequence of "hat" functions with support on $[0,1/n]$ and which has constant area, e.g. define $f_n(x) : [0,1] \to \mathbb{R}$

$$f_n(x) = \begin{cases} 0 \text{ if } x = 0 \\ n \text{ if } x = 1/(2n) \\ 0 \text{ if } 1/n \leq x \leq 1 \end{cases}$$

and interpolated linearly between $x = 0, 1/\left(2n\right), 1/n$. Then pointwise $\lim_{n\to\infty} f_n(x) = 0$ for any $x \in [0,1]$ (since $f_n(x) = 0$ if $x = 0$ or $n > 1/x$), but $\int_0^1 f_n(x)\, dx = 1/2$ for all $n$. So the answer is No.

6. Consider the function defined by the power series $\sum_{k=1}^{\infty} x^k/k$.

   (a) Find the set of all $x$ where the series converges.

   (b) Prove that the series converges uniformly on every interval $[-a, a]$, where $0 < a < 1$.

   (c) Find the sum of the series at every $x$ where it converges.

Show your work and justify all answers.

**Solution**

(a) The radius of convergence is

$$R = \frac{1}{\alpha}, \quad \alpha = \limsup_{k\to\infty} \sqrt[k]{\frac{1}{k}} = 1,$$

so the series converges for $-1 < x < 1$. Testing the endpoints individually, the series converges for $x = -1$ (alternating harmonic series), but not for $x = 1$ (harmonic series). Thus the series converges pointwise for $-1 \le x < 1$.

(b) Let $0 < a < 1$ and note that the series $\sum_{k=1} a^k/k$ converges provided $0 < a < 1$ (by comparison with the geometric series). Now note that $|x^k/k| \le a^k/k$ for $-a \le x \le a$. Thus the terms of the given series are bounded by the terms of a convergent series of constants. By the Weierstrass M-Test, the given series converges uniformly on $[-a, a]$.

Acceptable answer: it is known that if a power series centered at 0 converges with radius $R$, then it converges uniformly on every interval $[-a, a]$, $a < R$.

(c) It is known that if $f_n' \rightrightarrows g$ and $f_n$ converges at least at one point, then $f_n \rightrightarrows f$ such that $g = f'$, which applies to $f_n$ being partial sums of the series because the series for $f'$ converges with the same radius of convergence and thus uniformly in any interval $[-a, a]$, $a < 1$, and the series for $f$ converges e.g. at 0. Therefore, letting $f(x) = \sum_{k=1} x^k/k$, we have

$$f'(x) = \sum_{k=1} x^{k-1} = \sum_{k=0} x^k = \frac{1}{1-x},$$

on every interval $(-a, a)$, $a < 1$, and hence for all $x \in (-1, 1)$. Since $f(0) = 0$, we have for all $x \in (-1, 1)$,

$$f(x) = \int_0^x f'(t)\, dt = \int_0^x \frac{1}{1-x} dx = -\ln(1-x).$$

(There is also an alternative form of this part of the solution by integrating rather than differentiating a power series. Another acceptable answer: It is known that a power series can be differentiated term-by-term inside the interval of convergence.)

By Abel's theorem, since the series converges at $-1$, we have also

$$f(-1) = \lim_{x \to -1_+} -\ln(1-x) = -\ln 2.$$

The use of Abel's theorem is essential here; without it, nothing can be said about the value at $-1$.

7. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\begin{aligned} f(x,y) &= \frac{2xy}{4x^2+y^2} \text{ if } (x,y) \neq (0,0)\,, \\ f(0,0) &= 0. \end{aligned}$$

   (a) Evaluate $\lim_{(x,y)\to(0,0)} f(x,y)$.

   (b) Is $f$ continuous at $(0,0)$? Give a full explanation.

**Solution**

(a) In order for the limit to exist, $f(x,y)$ must approach the same value as $(x,y) \to (0,0)$ *from all directions*. Letting $(x,y) \to (0,0)$ with $y = mx$ (non-vertical lines), where $m$ is any finite constant, we find that $f(x,y) \to 2m/\left(4m^2+1\right)$, which clearly varies with $m$. Thus the limit does not exist.

(b) The function $f$ is not continuous at $(0,0)$ because $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

8. Show that the system of equations

$$\begin{aligned} x^2 z + y - z &= 0 \\ e^x + z &= 1 \end{aligned}$$

defines $x = x(z)$ and $y = y(z)$ uniquely as functions of $z$ in a neighborhood of the point $(x, y, z) = (0, 0, 0)$, and compute $\frac{\partial x}{\partial z}$and $\frac{\partial y}{\partial z}$ at $z = 0$.

**Solution**

Write the system of equations as

$$F(x, y, z) = 0,$$

where $F : \mathbb{R}^2 \times \mathbb{R}^1 \rightarrow \mathbb{R}^2$ is given by

$$F(x, y, z) = \begin{bmatrix} F_x \\ F_y \end{bmatrix} = \begin{bmatrix} x^2 z + y - z \\ e^x + z - 1 \end{bmatrix}.$$

Then $F(0, 0, 0) = (0, 0)$, all partial derivatives of $F$ are continuous, and the Jacobian with respect to $x$ and $y$ is

$$J = \det \begin{bmatrix} \frac{\partial F_x}{\partial x} & \frac{\partial F_x}{\partial y} \\ \frac{\partial F_y}{\partial x} & \frac{\partial F_y}{\partial y} \end{bmatrix} = \det \begin{bmatrix} 2xz & 1 \\ e^x & 0 \end{bmatrix} = \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1 \neq 0$$

at $(x, y) = (0, 0)$. Hence, the implicit function theorem applies, and the desired functions $x(z)$ and $y(z)$ exist and are differentiable. By the chain rule,

$$\frac{d}{dz} F(x(z), y(z), z) = \begin{bmatrix} \frac{\partial F_x}{\partial x} & \frac{\partial F_x}{\partial y} \\ \frac{\partial F_y}{\partial x} & \frac{\partial F_y}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial z} \end{bmatrix} + \begin{bmatrix} \frac{\partial F_x}{\partial z} \\ \frac{\partial F_z}{\partial z} \end{bmatrix} = 0,$$

so

$$\begin{bmatrix} \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial z} \end{bmatrix} = - \begin{bmatrix} \frac{\partial F_x}{\partial x} & \frac{\partial F_x}{\partial y} \\ \frac{\partial F_y}{\partial x} & \frac{\partial F_y}{\partial y} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial F_x}{\partial z} \\ \frac{\partial F_y}{\partial z} \end{bmatrix},$$

which at $(x, y, z) = (0, 0, 0)$ becomes

$$\begin{bmatrix} \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial z} \end{bmatrix} = - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

so at $z = 0$,

$$\frac{\partial x}{\partial z} = -1, \quad \frac{\partial y}{\partial z} = 1.$$

An alternative solution is possible by applying the inverse function theorem to the system

$$\begin{aligned} x^2 z + y &= a \\ e^x - 1 &= b \end{aligned}$$

and then the composition of the resulting inverse function with the function

$$z \mapsto \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -z \\ z \end{bmatrix}.$$

Note that here the symbols $x$ and $y$ sometimes mean variables and sometimes mean functions. This is an unfortunate but customary abuse of notation. A more careful solution can be written introducing extra notation for the functions, such as $X(z)$ and $Y(z)$.