University of Colorado at Denver — Mathematics Department

Applied Analysis Preliminary Exam

June 2, 2006

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write legibly using a dark pencil or pen.
- Notation: $\mathbb{R}$ denotes the set of real numbers; $\mathbb{Z}$ denotes the set of integers; and, $\mathbb{C}$ denotes the set of complex numbers. These extend to vector spaces as $\mathbb{R}^n$, $\mathbb{Z}^n$, and $\mathbb{C}^n$, respectively. Other notation will be defined as needed.
- Ask the proctor if you have any questions.

Good luck!

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________	8.	________
		Total	________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Analysis Preliminary Exam Committee:
Lynn Bennethum, Bill Briggs, Jan Mandel (Chair)

1. Show that if $A \subset \mathbb{R}$ is open then A has no isolated points. Show that this statement is not true in a general metric space, i.e., give an example of a metric space (X, ρ), where there exists an open set $A \subset X$ which has an isolated point.

2. Prove that there is no continuous map from the closed interval $[0, 1]$ onto the open interval $(0, 1)$. Hint: if f is such a function, then there exists $x_n \in f^{-1}((0, 1/n))$.

3. Let $\{a_k\}$ be a sequence in $\mathbb{R}$. Using the definitions, prove the following in $\mathbb{R}$: $\sup\{a_k : k \in N\}$ does not exist $\iff$ $\{a_k\}$ is not bounded above $\iff$ $\forall n \in \mathbb{N}$: $\sup\{a_k : k \geq n\}$ does not exist

4. Let $f_n : \mathbb{R} \to \mathbb{R}$ be differentiable for each $n = 1, 2, ...$ with $|f_n'(x)| \leq 1$ for all n and for all x. Assume
$$\lim_{n\to\infty} f_n(x) = g(x) \qquad \forall x \in \mathbb{R}.$$
Prove that $g : \mathbb{R} \to \mathbb{R}$ is continuous.

5. Let $f_n(x) : [0,1] \to \mathbb{R}$ be a sequence of continuous functions. Suppose that $\lim_{n\to\infty} f_n(x) = 0$ for each $x \in [0,1]$ and also that, for some constant K, we have

$$\left| \int_0^1 f_n(x)\, dx \right| \leq K < \infty \qquad \forall n.$$

Does it hold that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = 0?$$

Prove or give a counterexample.

6. Consider the function defined by the power series $\sum_{k=1}^{\infty} x^k/k$.

   (a) Find the set of all x where the series converges.

   (b) Prove that the series converges uniformly on every interval $[-a, a]$, where $0 < a < 1$.

   (c) Find the sum of the series at every x where it converges.

   Show your work and justify all answers.

7. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\begin{aligned} f(x,y) &= \frac{2xy}{4x^2+y^2} \text{ if } (x,y) \neq (0,0)\,, \\ f(0,0) &= 0. \end{aligned}$$

   (a) Evaluate $\lim_{(x,y)\to(0,0)} f(x,y)$.

   (b) Is f continuous at $(0,0)$? Give a full explanation.

8. Show that the system of equations

$$\begin{aligned} x^2 z + y - z &= 0 \\ e^x + z &= 1 \end{aligned}$$

defines $x = x(z)$ and $y = y(z)$ uniquely as functions of z in a neighborhood of the point $(x, y, z) = (0, 0, 0)$, and compute $\frac{\partial x}{\partial z}$ and $\frac{\partial y}{\partial z}$ at $z = 0$.