

University of Colorado at Denver — Mathematics Department

Applied Analysis Preliminary Exam

January 14, 2008

Name: _____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write legibly using a dark pencil or pen.
- Notation: \mathbb{R} denotes the set of real numbers; \mathbb{Z} denotes the set of integers; and, \mathbb{C} denotes the set of complex numbers. These extend to vector spaces as \mathbb{R}^n , \mathbb{Z}^n , and \mathbb{C}^n , respectively. Other notation will be defined as needed.
- Ask the proctor if you have any questions.

Good luck!

1. _____	5. _____
2. _____	6. _____
3. _____	7. _____
4. _____	8. _____

Total _____

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Analysis Preliminary Exam Committee:

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1. Let A be a set of reals that is bounded above, and $B \subset A$ nonempty. Prove from the definition of supremum that $\sup B \leq \sup A$.

2. Let

$$A = \{ f \in C([0, 1]) : \forall x \in [0, 1], f(x) \neq 0 \},$$

where $C([0, 1])$ is the space of all continuous functions from $[0, 1]$ to \mathbb{R} , equipped with the norm

$$\|f\|_{\infty} = \max \{|f(x)| : x \in [0, 1]\}.$$

Show that the set A is an open subset of $C([0, 1])$.

3. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a uniformly continuous function. Prove that there exist $a > 0$ and $b > 0$ such that

$$\forall t \in \mathbb{R}, \quad |f(t)| \leq a \cdot |t| + b.$$

4. Let f be differentiable on $[a, b]$, with $|f'| \leq c$. Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$. Prove that

$$\left| \int_a^b f - \sum_{i=1}^n f(x_i)(x_i - x_{i-1}) \right| \leq \frac{c}{2} \|P\| (b - a),$$

where $\|P\| = \max_{i=1, \dots, n} (x_i - x_{i-1})$ is the largest gap in P .

5. Suppose that the function $f : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable at x_0 . Prove that

$$\lim_{x \rightarrow x_0} \frac{xf(x_0) - x_0f(x)}{x - x_0} = f(x_0) - x_0f'(x_0).$$

6. Let $f : [0, +\infty) \rightarrow \mathbb{R}$ be a continuous function such that $f(0) = 0$ and $\lim_{x \rightarrow +\infty} f(x) = 0$. Let

$$g_n(x) = f(nx)f\left(\frac{x}{n}\right).$$

Prove that g_n converges uniformly to 0 on $[0, +\infty)$. (Hint: first prove that f is bounded on $[0, +\infty)$.)

7. Prove that

$$\frac{1}{(1+x)^2} = \sum_{k=1}^{\infty} (-1)^{k-1} kx^{k-1} \quad \forall x \in (-1, 1).$$

8. Find the derivative at the point 0 of the function $y = y(x)$ defined by the equation

$$\ln(x^2 + y^2 + 1) = y$$

in a neighborhood of the point $(x, y) = (0, 0)$.