University of Colorado at Denver — Mathematics Department

Applied Analysis Preliminary Exam

January 9, 2006

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write legibly using a dark pencil or pen.
- Notation: $\mathbb{R}$ denotes the set of real numbers; $\mathbb{Z}$ denotes the set of integers; and, $\mathbb{C}$ denotes the set of complex numbers. These extend to vector spaces as $\mathbb{R}^n$, $\mathbb{Z}^n$, and $\mathbb{C}^n$, respectively. Other notation will be defined as needed.
- Ask the proctor if you have any questions.

Good luck!

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________	8.	________
		Total	________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Analysis Preliminary Exam Committee:
Lynn Bennethum, Bill Briggs, Jan Mandel (Chair)

1. Let $K \subset \mathbb{R}$ consist of 0 and the numbers $1/n$, for $n = 1, 2, \ldots$. Prove that K is compact from the definition, without the use of the Heine-Borel theorem.

2. Define $\{a_n\}$ recursively by

$$a_1 = \sqrt{k}; \qquad a_{n+1} = \sqrt{k + a_n}, \quad (n \geq 1),$$

where k is a fixed constant, $k > 1$.

(a) (16 points) Prove the sequence $\{a_n\}$ has a limit.

(b) (4 points) Find the limit.

Justify all steps.

3. Consider the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, with the property that

$$0 < \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = \ell < \infty.$$

   (a) What is the largest value of r *implied by this condition* such that the power series converges absolutely for $|x-a| < r$?

   (b) Let $r_1 < r$. Prove that the power series given above converges uniformly on the set $\{x : |x-a| \leq r_1\}$.

4. Consider the sequence of functions $f_n(x) = 2(n+1)x(1-x^2)^n$ for $x \in [0,1]$.

   (a) (10 points) What is the pointwise limit of $f_n(x)$?

   (b) (5 points) What is $\int_0^1 f_n(x)dx$?

   (c) (5 points) Does the sequence of functions converge uniformly to its pointwise limit on the interval $[0,1]$? Justify all steps.

5. (a) (5 points) Suppose f is continuous on the interval $I = [a, b]$. Does f necessarily have a fixed point on I if $|f'(x)| < 1$ for all $x \in I$? Explain.

   (b) (3 points) Find all real fixed points x^* of the function $f(x) = x^3 + x - 8$.

   (c) (12 points) With f given above, find all values of x_0 for which the iteration $x_{n+1} = f(x_n)$ for $n = 0, 1, 2, \ldots$ converges. Justify the answer. If any theorems are used, identify them clearly; if convenient theorems are not available, work out a detailed proof.

6. Suppose $|f(x)| \leq 1$ for all $x \in [a, b]$, f is continuous on $[a, b]$, and $g(x)$ is monotonically increasing on $[a, b]$. Show that

$$\left| \int_a^b f dg \right| \leq g(b) - g(a).$$

7. Let $x = f(u, v) = u^2 - v^2$ and $y = g(u, v) = 2uv$.

   (a) Find the Jacobian of the transformation $T : (u, v) \mapsto (x, y)$.

   (b) Express $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ in terms of u and v.

   (c) Does the transformation $T : (u, v) \mapsto (x, y)$ have an inverse in the neighborhood of the point $(u, v) = (2, -1)$? Prove your answer.