

University of Colorado at Denver — Mathematics Department

Applied Analysis Preliminary Exam

August 27, 2005

Name: _____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write legibly using a dark pencil or pen.
- Notation: \mathbb{R} denotes the set of real numbers; \mathbb{Z} denotes the set of integers; and, \mathbb{C} denotes the set of complex numbers. These extend to vector spaces as \mathbb{R}^n , \mathbb{Z}^n , and \mathbb{C}^n , respectively. Other notation will be defined as needed.
- Ask the proctor if you have any questions.

Good luck!

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| 2. _____ | 6. _____ |
| 3. _____ | 7. _____ |
| 4. _____ | 8. _____ |

Total _____

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Analysis Preliminary Exam Committee:

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1. Let (A, d) be a metric space, $K \subset A$ a nonempty compact set, and $x \in A \setminus K$. Show that there exists $y \in K$ such that $d(x, y)$ is minimal.

Solution

Define $f : K \rightarrow \mathfrak{R}$, $f(y) = d(x, y)$, the distance between x and y . Claim: f is a continuous function. Pf: Let $\epsilon > 0$ be given and let $\delta = \epsilon$. Then $\forall y_1, y_2 \in K$ such that $d(y_1, y_2) < \delta$ we have $d(f(y_1), f(y_2)) = |d(x, y_1) - d(x, y_2)|$. Now note that $|d(x, y_1) - d(x, y_2)| \leq d(y_1, y_2)$, which we can show as follows:

$$\begin{aligned} d(x, y_1) &\leq d(y_1, y_2) + d(x, y_2) \\ d(x, y_1) - d(x, y_2) &\leq d(y_1, y_2) \end{aligned}$$

Similarly,

$$d(x, y_2) - d(x, y_1) \leq d(y_1, y_2).$$

Thus $|d(x, y_1) - d(x, y_2)| \leq d(y_1, y_2) < \delta = \epsilon$ and we've shown that f is continuous. Since f is a continuous function on a compact set, the function achieves a maximum and minimum value. Thus there exists $y \in K$ such that $f(y) = d(x, y)$ is minimal.

Alternate Solution

Let $a = \inf \{d(x, z) : z \in K\}$. From the definition of infimum, for every n , there is $y_n \in K$ such that $d(x, y_n) < a + 1/n$. Because K is compact, it is sequentially compact, so the sequence y_n has a convergent subsequence y_{n_k} with limit $y \in K$, and $d(y_{n_k}, y) \rightarrow 0$ as $k \rightarrow \infty$. Using the definition of a and the triangle inequality,

$$a \leq d(x, y) \leq d(x, y_{n_k}) + d(y_{n_k}, y) < a + \frac{1}{n_k} + d(y_{n_k}, y) \rightarrow a, \quad k \rightarrow \infty,$$

It follows that $d(x, y) = a$.

2. Let $f : D \subset \mathbb{R} \rightarrow \mathbb{R}$ and $x \in D$. Prove that f is continuous at x (using the $\varepsilon - \delta$ definition) if and only if for every sequence $\{x_n\}$, $x_n \in D \forall n$, $x_n \rightarrow x$ implies $f(x_n) \rightarrow f(x)$.

Solution

(i) Suppose that f is continuous at x and $x_n \rightarrow x$. Let $\varepsilon > 0$. Since f is continuous at x , there is $\delta > 0$ such that $\forall x \in D : |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$. Since $x_n \rightarrow x$, there is N such that $n > N \implies |x_n - x| < \delta$. So, $n > N \implies |f(x_n) - f(x)| < \varepsilon$.

(ii) Suppose that f is not continuous at x ; by negation of the definition of continuity, this means that $\exists \varepsilon > 0 \forall \delta > 0 \exists y \in D : |y - x| < \delta$ and $|f(y) - f(x)| \geq \varepsilon$. Choosing $\delta = 1/n$ in turn gives a sequence x_n such that $|x_n - x| < 1/n$ and $|f(x_n) - f(x)| \geq \varepsilon$, so $x_n \rightarrow x$ and $\neg(f(x_n) \rightarrow f(x))$.

3. Consider the function defined by $f(x) = \sin \frac{1}{x}$, $x > 0$, $f(0) = 0$. Using the definition of Riemann integral, show that f is Riemann integrable on the interval $[0, 1]$.

Solution

Given $\varepsilon > 0$, we need to find a partition P of $[0, 1]$, such that $U(f, P) - L(f, P) < c\varepsilon$, for some constant c . On the compact interval $[\varepsilon, 1]$, the function f is continuous, hence uniformly continuous. Let $\delta > 0$ be such that $\delta < \varepsilon$ and

$$\forall x, y \in [\varepsilon, 1] : |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

Let $P = [x_0 = 0, x_1, \dots, x_n = 1]$ be a partition of $[0, 1]$ with $|P| = \max(x_{i+1} - x_i) < \delta$ and $k = \min\{i : x_i > \varepsilon\}$. Then

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^n \left(\sup_{[x_{i-1}, x_i]} f \right) (x_i - x_{i-1}) - \sum_{i=1}^n \left(\inf_{[x_{i-1}, x_i]} f \right) (x_i - x_{i-1}) \\ &= \sum_{i=1}^k \left(\sup_{[x_{i-1}, x_i]} f - \inf_{[x_{i-1}, x_i]} f \right) (x_i - x_{i-1}) \\ &\quad + \sum_{i=k+1}^n \left(\sup_{[x_{i-1}, x_i]} f - \inf_{[x_{i-1}, x_i]} f \right) (x_i - x_{i-1}) \end{aligned}$$

Estimate the first term by boundedness of f and $x_{k-1} \leq \varepsilon$ from the definition of k ,

$$\begin{aligned} \sum_{i=1}^k \left(\sup_{[x_{i-1}, x_i]} f - \inf_{[x_{i-1}, x_i]} f \right) (x_i - x_{i-1}) &\leq 2 \sum_{i=1}^k (x_i - x_{i-1}) \\ &= 2(x_{k-1} - x_0) + 2(x_k - x_{k-1}) \\ &= 2\varepsilon + 2\delta \leq 4\varepsilon \end{aligned}$$

and the second term from uniform continuity of f using $x_k > \varepsilon$: since f is continuous on the compact interval $[x_{i-1}, x_i]$,

$$\inf_{[x_{i-1}, x_i]} f = f(c_i), c_i \in [x_{i-1}, x_i], \quad \sup_{[x_{i-1}, x_i]} f = f(d_i), d_i \in [x_{i-1}, x_i]$$

and since $|c_i - d_i| < \delta$, we have

$$\sum_{i=k+1}^n \left(\sup_{[x_{i-1}, x_i]} f - \inf_{[x_{i-1}, x_i]} f \right) (x_i - x_{i-1}) \leq \sum_{i=k+1}^n \varepsilon (x_i - x_{i-1}) \leq \varepsilon$$

Putting the two estimates together,

$$U(f, P) - L(f, P) \leq 4\varepsilon + \varepsilon < 6\varepsilon$$

4. Prove that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$.

Solution

Define $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$ and $g(x) = \sum_{n=1}^{\infty} (-1)^{n+1} x^{n-1} = \sum_{k=0}^{\infty} (-1)^k x^k$. Since the radius of convergence of the series for g is $R = 1$, and g is term by term derivative of f , both series converge uniformly on any interval $[-a, a]$, $a < R = 1$, and $f'(x) = g(x)$ for every $x \in (-1, 1)$. By the sum of geometric series,

$$g(x) = \frac{1}{1+x}$$

and using $f(0) = 0$, we get

$$f(x) = \int_0^x g(t) dt = \ln(1+x) \quad \forall x \in (-1, 1)$$

The series for $f(x)$ converges by the alternating series test also for $x = 1$. Consequently, from Abel's theorem, the series for $f(x)$ converges uniformly on any interval $[-a, 1]$, $a < 1$. From this and continuity of the \ln function,

$$\ln 2 = \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = \sum_{n=1}^{\infty} \lim_{x \rightarrow 1^+} \frac{(-1)^{n+1}}{n} x^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}.$$

Note: Use of Abel's theorem is essential. This is the whole point of the problem. It does not follow from the fact that the radius of convergence of a series $\sum_{n=0}^{\infty} a_n x^n$ is $R = 1$ that it converges uniformly on $(-1, 1)$ or on $(0, 1)$. Abel's theorem is required to conclude from $R = 1$ and the fact that the series converges also for $x = 1$ that convergence is in fact uniform on $(0, 1]$ or $(0, 1)$ (which is the same thing, but to show that is a little exercise in itself). Sometimes the conclusion of Abel's theorem is formulated as $\lim_{x \rightarrow 1^+} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n$ without mentioning uniform convergence, so this is also acceptable.

5. Let $\{a_n\}$ be a monotonically decreasing sequence of real numbers $a_n > 0$ such that $\sum_{n=1}^{\infty} a_n$ converges. Prove that $na_n \rightarrow 0$, $n \rightarrow \infty$.

Solution

Suppose $na_n \not\rightarrow 0$, $n \rightarrow \infty$. Then there exists $\varepsilon > 0$ such that for every M there is $n > M$ such that $|na_n| \geq \varepsilon$. Since $a_n \geq 0$, this implies $na_n \geq \varepsilon$. So, there exists $n_1 > 0$ such that $n_1 a_{n_1} \geq \varepsilon$, and by induction construct sequence $n_k > 2n_{k-1}$ such that $n_k a_{n_k} \geq \varepsilon$. Since $\{a_n\}$ is monotonically decreasing, we have

$$\sum_{n=n_{k-1}+1}^{n_k} a_n \geq (n_k - n_{k-1}) a_{n_k} \geq \frac{1}{2} n_k a_{n_k} \geq \frac{1}{2} \varepsilon,$$

which also holds for $k = 1$ if we put $n_0 = 0$. Consequently,

$$\sum_{n=1}^{n_k} a_n \geq \frac{k}{2} \varepsilon.$$

Hence, a subsequence of partial sums diverges to $+\infty$, which contradicts the assumption that $\sum_{n=1}^{\infty} a_n$ converges. (In fact the whole sequence of partial sums diverges to $+\infty$ but it is not necessary to say or show this.)

Alternate Solution

For any $n > m$, it holds that

$$(n - m)a_n \leq \sum_{k=m+1}^n a_k,$$

since $a_m \geq a_{m+1} \geq \dots \geq a_n$. Let $\varepsilon > 0$. From the Cauchy property of convergent series, there exists an $N > 0$ such that

$$n > m > N \implies \sum_{k=m+1}^n a_k < \varepsilon.$$

Now let $n > 2N + 1$. Then $n/2 \geq [n/2] > N$, so

$$na_n \leq 2(n - [n/2])a_n \leq 2 \sum_{k=[n/2]+1}^n a_k < 2\varepsilon.$$

6. Consider the sequence of functions $f_n(x) = n^2 x e^{-nx}$, where $n = 1, 2, \dots$, and $x \in [0, 1]$.

(a) Determine the pointwise limit, $f(x) = \lim_{n \rightarrow \infty} f_n(x)$.

(b) Calculate $\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx$.

(c) Calculate $\int_0^1 \lim_{n \rightarrow \infty} f_n(x) dx$.

(d) Explain your results, proving statements where appropriate.

Solution

(a) Using l'Hôpital's rule,

$$\begin{aligned} f(x) &= \lim_{n \rightarrow \infty} f_n(x) \\ &= \lim_{n \rightarrow \infty} \frac{n^2 x}{e^{nx}} \\ &= \lim_{n \rightarrow \infty} \frac{2n}{e^{nx}} \\ &= \lim_{n \rightarrow \infty} \frac{2}{x e^{nx}} \\ &= 0 \end{aligned}$$

(b) Using integration by parts,

$$\int_0^1 f_n(x) dx = n^2 \left[-\frac{x}{n} e^{-nx} - \frac{e^{-nx}}{n^2} \right]_0^1 = -n e^{-n} - e^{-n} + 1.$$

$$\text{So } \lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx = 1.$$

$$(c) \int_0^1 \lim_{n \rightarrow \infty} f_n(x) dx = \int_0^1 0 dx = 0.$$

(d) The different answers given by parts (b) and (c) imply that the sequence of functions f_n does not converge uniformly since the integral of the limit of uniformly convergent sequence of integrable functions is the limit of their integrals. (It is not necessary to prove that directly). Another answer: From the dominated convergence theorem, it follows that there is no integrable function g such that $|f_n(x)| \leq g(x)$ a.e. on $(0, 1)$ for all n . This is a stronger conclusion, but it is outside of the scope of the syllabus.

7. Show that a compact metric space has a countable dense subset.

Solution

Let X be a compact metric space. Given any $n \in \mathbb{N}$, the balls $B(x, 1/n)$, $\forall x \in X$, form an open cover of X . Since X is compact, there is a finite subcover

$$X \subset \bigcup_{i=1}^{m_n} B(x_{i,n}, 1/n).$$

Let A be the set of all centers of such balls:

$$A = \{x_{i,n} : i = 1, \dots, m_n, n \in \mathbb{N}\}.$$

Since A is a countable union of finite sets, it is countable. We will prove that A is dense in X . Let $x \in X$ and $\varepsilon > 0$. Choose $n \in \mathbb{N}$ such that $1/n < \varepsilon$. Since $X \subset \bigcup_{i=1}^{m_n} B(x_{i,n}, 1/n)$, it holds $x \in B(x_{i,n}, 1/n)$ for some i . Therefore $d(x, x_{i,n}) < 1/n < \varepsilon$ with $x_{i,n} \in A$.

8. (Implicit Function Theorem in 1D) Consider $F(x, y) : \mathbb{R}^2 \rightarrow \mathbb{R}$ where $F \in C^1([x_0 - \delta, x_0 + \delta] \times [y_0 - \delta, y_0 + \delta])$. Assume that $F(x_0, y_0) = 0$ and that $F_y(x_0, y_0) = \frac{\partial F}{\partial y}(x_0, y_0) > 0$. Prove that there exists a unique function $f(x)$ such that $F(x, f(x)) = 0$ in a neighborhood about (x_0, y_0) .

Solution

Since F_y is continuous and increasing, there exists a neighborhood about (x_0, y_0) such that $F_y(x, y) > 0$. Let $[x_0 - \delta_1, x_0 + \delta_1] \times [y_0 - \delta_1, y_0 + \delta_1]$ be contained in this neighborhood. Hence for each fixed x in this δ_1 neighborhood, $F(x, y)$ is a strictly increasing function for $y \in [y_0 - \delta_1, y_0 + \delta_1]$ and

$$F(x_0, y_0 + \delta_1) > F(x_0, y_0) = 0, \quad F(x_0, y_0 - \delta_1) < F(x_0, y_0) = 0.$$

By the continuity of $F(x, y_0 + \delta_1)$ and $F(x, y_0 - \delta_1)$ there exists a neighborhood about x_0 such that

$$F(x, y_0 + \delta_1) > 0, \quad F(x, y_0 - \delta_1) < 0, \quad x \in [x_0 - \eta, x_0 + \eta],$$

where $0 < \eta < \delta_1$. For each $x \in [x_0 - \eta, x_0 + \eta]$, $F(x, y)$ is a continuous function, so there exists just one value of $y \in [y_0 - \delta_1, y_0 + \delta_1]$ such that $F(x, y) = 0$. Let $f(x) = y$. Note that there cannot be two such values of y because that would imply F_y would be zero, which contradicts $F_y > 0$ in the neighborhood defined by δ_1 (Rolle's theorem).