

University of Colorado at Denver — Mathematics Department

Applied Linear Algebra Preliminary Exam With Solutions

1 June 2009, 10:00 am – 2:00 pm

Name: \_\_\_\_\_

The proctor will let you read the following conditions before the exam begins, and you will have time for questions. Once the exam begins, you will have 4 hours to do your best. This is a closed book exam. Please put your name on each sheet of paper that you turn in.

**Exam conditions:**

- Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value unless stated otherwise..
- Justify your solutions: cite theorems that you use, provide counter-examples for dis-proof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write legibly using a dark pencil or pen.
- Notation:  $\mathcal{C}$  denotes the field of complex numbers,  $\mathbb{R}$  denotes the field of real numbers, and  $F$  denotes a field which may be either  $\mathcal{C}$  or  $\mathbb{R}$ .  $\mathcal{C}^n$  and  $\mathbb{R}^n$  denote the vector spaces of  $n$ -tuples of complex and real scalars, respectively, written as column vectors. For  $T \in \mathcal{L}(V)$ , the *image* (sometimes called the *range*) of  $T$  is denoted  $\text{Im}(T)$ . If  $A$  is a matrix over a field, then  $\text{rk}(A)$  is the *rank of*  $A$ . For  $x \in \mathbb{R}^n$ ,  $\|\cdot\|$  denotes the usual Euclidean norm. If  $A$  is an  $m \times n$  matrix over  $F$ ,  $T_A$  is the linear map defined by

$$T_A: F^n \rightarrow F^m: x \mapsto Ax.$$

$T^*$  is the adjoint of the operator  $T$  and  $\lambda^*$  is the complex conjugate of the scalar  $\lambda$ .  $v^T$  and  $A^T$  denote vector and matrix transposes, respectively.

- Ask the proctor if you have any questions.

Good luck!

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| 1. _____ | 5. _____ |
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| 4. _____ | 8. _____ |

Total \_\_\_\_\_

1. Let  $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ . Define  $T : M_2(\mathbb{R}) \rightarrow M_2(\mathbb{R})$  by

$$T: B \mapsto AB - BA.$$

- (i) (8 points) Fix an ordered basis  $\mathcal{B}$  of  $M_2(\mathbb{R})$  and compute the matrix  $[T]_{\mathcal{B}}$  that represents  $T$  with respect to this basis.
- (ii) (8 points) Compute a basis for each of the eigenspaces of  $T$ .
- (iii) (4 points) Give the minimal and characteristic polynomials of  $T$  and the Jordan form for  $T$ .

**Solution:** We choose the "standard ordered basis"

$$\mathcal{B} = (E_{11}, E_{12}, E_{21}, E_{22})$$

where  $E_{ij}$  has a 1 in the  $(i, j)$  position and zero elsewhere.

Then routine computations show that

$$T: E_{11} \mapsto 2(E_{21} - E_{12})$$

$$T: E_{12} \mapsto 2(E_{22} - E_{11})$$

$$T: E_{21} \mapsto 2(E_{11} - E_{22})$$

$$T: E_{22} \mapsto 2(E_{12} - E_{21})$$

From this it is easy to write down the matrix  $[T]_{\mathcal{B}}$ , and then write down

$$\lambda I - [T]_{\mathcal{B}} = \begin{pmatrix} \lambda & 2 & -2 & 0 \\ 2 & \lambda & 0 & -2 \\ -2 & 0 & \lambda & 2 \\ 0 & -2 & 2 & \lambda \end{pmatrix}.$$

In a few steps we evaluate the determinant of this matrix and find that the characteristic polynomial of  $T$  is

$$f(\lambda) = \lambda^2(\lambda - 4)(\lambda + 4).$$

Put  $\lambda = 0$  in the matrix and row reduce to find that the null space has the basis  $(E_{11} + E_{22}, E_{12} + E_{21})$

A basis for the eigenspace with  $\lambda = 4$  is

$$(E_{11} - E_{12} + E_{21} - E_{22}).$$

A basis for the eigenspace with  $\lambda = -4$  is

$$(E_{11} + E_{12} - E_{21} - E_{22}).$$

So the Jordan form is a diagonal matrix with diagonal entries  $0, 0, 4, -4$ , and the minimal polynomial is

$$p(\lambda) = \lambda(\lambda - 4)(\lambda + 4).$$

2. Let  $A \in M_6(\mathbb{C})$  be defined by

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

Find all of the eigenvalues, eigenvectors, and generalized eigenvectors of  $A$ . Construct the characteristic polynomial, the minimal polynomial, and the Jordan form of  $A$ .

**Solution:** The characteristic polynomial is  $x^2(x + 1)^4$  and the eigenvalues are  $0$  and  $-1$ . The eigenvectors associated with eigenvalue  $0$  are of the form  $[0, a, 0, 0, 0, 0]^T$  and the generalized eigenvectors are of the form  $[a, b, 0, 0, 0, 0]^T$ . The eigenvectors associated with  $-1$  are of the form  $[0, 0, 0, a, b, -a]^T$ , and the generalized eigenvectors are of the form  $[0, 0, 0, a, b, c]^T$ . The minimal polynomial is  $x^2(x + 1)^3$ . One Jordan form is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

3. Norms of Linear Operators

(a) Let  $A$  be an  $m \times n$  real matrix. Prove that there is a real constant  $M_A$  such that  $\|A\mathbf{x}\| \leq M_A\|\mathbf{x}\|$  for all  $\mathbf{x} \in \mathbb{R}^n$ .

**Solution:** We use two basic facts: First,  $\|(x_1, \dots, x_m)\| \leq \sum_{i=1}^m |x_i|$ , by the triangle inequality; Second, recall the Cauchy-Schwartz inequality:  $|\mathbf{x} \cdot \mathbf{v}| \leq \|\mathbf{x}\| \cdot \|\mathbf{v}\|$ . So we have:

$$\begin{aligned} \|A\mathbf{v}\| &= \|(a_{ij}) \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}\| = \left\| \begin{pmatrix} (a_{11}, \dots, a_{1n}) \cdot \mathbf{v} \\ \vdots \\ (a_{m1}, \dots, a_{mn}) \cdot \mathbf{v} \end{pmatrix} \right\| \leq \\ &\leq \sum_{i=1}^m \|(a_{i1}, \dots, a_{in}) \cdot \mathbf{v}\| \leq \sum_{i=1}^m \|(a_{i1}, \dots, a_{in})\| \cdot \|\mathbf{v}\| = M_A\|\mathbf{v}\|, \end{aligned}$$

where  $M_A = \sum_{i=1}^m \sqrt{a_{i1}^2 + a_{i2}^2 + \dots + a_{in}^2}$ .

(b) Let  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map. Prove that there is some positive constant  $\|T\|$  for which

$$\|T(\mathbf{v})\| \leq \|T\| \cdot \|\mathbf{v}\|$$

for all  $\mathbf{v} \in \mathbb{R}^n$ .

**Solution:** If  $T = T_A: \mathbb{R}^n \rightarrow \mathbb{R}^m$  with  $A = (a_{ij})$ , then  $\|T(\mathbf{v})\| = \|Av\| \leq M_A \|\mathbf{v}\|$  (by part (a)) for all  $\mathbf{v} \in \mathbb{R}^n$ , where  $M_A = \sum_{i=1}^m \sqrt{a_{i1}^2 + a_{i2}^2 + \cdots + a_{in}^2} \in \mathbb{R}$ .

#### 4. Spheres in Finite Dimensional Real Vector Spaces

Let  $\mathcal{B} = (v_1, v_2, \dots, v_n)$  be an ordered basis of the real vector space  $V$  with dimension  $n$ . For each  $v \in V$  there are unique scalars  $c_1, \dots, c_n \in \mathbb{R}$  for which  $v = \sum_{i=1}^n c_i v_i$ . Write the *coordinate matrix*  $[v]_{\mathcal{B}}$  of  $v$  with respect to the ordered basis  $\mathcal{B}$  as

$$[v]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, \text{ so that } v = (v_1, \dots, v_n) \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \mathcal{B} \cdot [v]_{\mathcal{B}}.$$

For  $\mathbf{c} = [c_1, \dots, c_n]^T \in \mathbb{R}^n$ , we employ the usual Euclidean norm:

$$\|\mathbf{c}\| = \sqrt{\sum_{i=1}^n c_i^2}.$$

For an arbitrary ordered basis  $\mathcal{B}$  of  $V$ , we define the norm with respect to  $\mathcal{B}$  as follows:

$$\|v\|_{\mathcal{B}} := \|[v]_{\mathcal{B}}\|.$$

Given the basis  $\mathcal{B}$ , a specific vector  $v_0$  and a positive number  $r$  we can define the  $n$ -dimensional sphere with center  $v_0$  and radius  $r$  (with respect to  $\mathcal{B}$ ) by

$$S_{r, \mathcal{B}}(v_0) = \{w \in V : \|v_0 - w\|_{\mathcal{B}} \leq r\}.$$

**Problem** Let  $r > 0$  and let  $\mathcal{B}, \mathcal{B}'$  be any two ordered bases of  $V$ . Show that there is an  $r' > 0$  such that

$$S_{r, \mathcal{B}}(\mathbf{0}) \subseteq S_{r', \mathcal{B}'}(\mathbf{0}).$$

**Solution:** If  $\mathcal{B}$  and  $\mathcal{B}'$  are two given ordered bases of  $V$ , there is an invertible, real  $n \times n$  matrix  $A$  for which  $\mathcal{B}' = \mathcal{B}A$ , so that  $[v]_{\mathcal{B}} = A \cdot [v]_{\mathcal{B}'}$ . Then there is a constant  $\|A\|$  such that  $\|AX\| \leq \|A\| \cdot \|X\|$  for any  $X \in \mathbb{R}^n$ . Hence

$$\|v\|_{\mathcal{B}} = \|[v]_{\mathcal{B}}\| = \|A \cdot [v]_{\mathcal{B}'}\| \leq \|A\| \cdot \|[v]_{\mathcal{B}'}\| = \|A\| \cdot \|v\|_{\mathcal{B}'}$$

From this we see that

$$S_{r, \mathcal{B}A}(\mathbf{0}) \subseteq S_{r/\|A\|, \mathcal{B}}(\mathbf{0}). \tag{1}$$

Of course the argument is symmetric in  $\mathcal{B}$  and  $\mathcal{B}'$ .

## 5. Fredholm Alternative

Let  $A$  be an  $m \times n$  real matrix and  $b \in \mathbb{R}^m$ . Show that exactly one of the following systems has a solution:

- i)  $Ax = b$
- ii)  $A^T y = 0, \quad y^T b \neq 0.$

Note: Our notation is  $y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$ , so  $y^T = [y_1, \dots, y_m]$ .

**Solution:** If  $b \in \text{col}A$ , then statement i) has a solution, but since  $\text{col}A \perp \text{null}A^T$ , statement ii) has no solution.

If  $b \notin \text{col}A$ , then statement i) does not have a solution. In this case, let  $z = \text{proj}_{\text{col}A} b$  (the orthogonal projection of  $b$  onto the column space of  $A$ ), and define  $y = b - z$ . Note that  $y \neq 0$  (since  $b \notin \text{col}A$ ). Note also that since  $z$  is an orthogonal projection,  $y \in (\text{col}A)^\perp = \text{null}A^T$ . Thus,  $A^T y = 0$  and  $y^T b = y^T (y + z) = y^T y \neq 0$ , so statement ii) has a solution.

## 6. Upper-triangularization

(a) (12 points) For each of the following, if it is true, merely say so; if it is false, give a counterexample.

- (i) If  $V$  is a finite-dimensional vector space over  $\mathbb{R}$  and  $T \in \mathcal{L}(V)$ , then  $V$  has a basis  $\mathcal{B}$  with respect to which  $[T]_{\mathcal{B}}$  is upper triangular.

**Solution: FALSE** Suppose  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2: (x, y) \mapsto (y, -x)$ . Then if  $\mathcal{S}$  is the standard ordered basis of  $\mathbb{R}^2$ , the matrix

$$[T]_{\mathcal{S}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

so that the characteristic polynomial of  $T$  is  $x^2 + 1$ . This polynomial has no real roots, so  $T$  has no real eigenvalues, which would have to lie along the diagonal of  $[T]_{\mathcal{B}}$  if  $V$  had such a basis.

- (ii) If  $V$  is a finite-dimensional vector space over  $\mathcal{C}$  and  $T \in \mathcal{L}(V)$ , then  $V$  has a basis  $\mathcal{B}$  with respect to which  $[T]_{\mathcal{B}}$  is upper triangular.

**Solution: TRUE** (This is usually called the Theorem of Schur.)

- (iii) If  $V$  is a finite-dimensional vector space over  $\mathcal{C}$  and  $S, T \in \mathcal{L}(V)$ , then  $V$  has a basis  $\mathcal{B}$  for which both  $[S]_{\mathcal{B}}$  and  $[T]_{\mathcal{B}}$  are upper triangular.

**Solution: False** Suppose that with respect to some basis  $\mathcal{B}'$ ,  $S$  and  $T$  have the following matrices:

$$[S]_{\mathcal{B}'} = \begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix} \quad \text{and} \quad [T]_{\mathcal{B}'} = \begin{pmatrix} 0 & 1 \\ d & 0 \end{pmatrix},$$

with  $c \neq d$ . Then a basis  $\mathcal{B}$  of the desired type would exist if and only if there were an invertible matrix  $P = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$  for which

$$\begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix} \begin{pmatrix} h & -f \\ -g & e \end{pmatrix} = \begin{pmatrix} * & * \\ ch^2 - g^2 & * \end{pmatrix},$$

with a similar equation holding for the other matrix. It follows that there would have to be an invertible matrix  $P$  as above with  $g^2 = ch^2$  and  $g^2 = dh^2$ . If  $g = 0$ , then  $h \neq 0$ , implying that  $c = 0$  and  $d = 0$ , contradicting  $c \neq d$ .

(b) (8 points) Show that a normal, upper triangular matrix must be diagonal.

**Solution:** We may assume that  $A$  is  $n \times n$  with entries in  $\mathcal{C}$ , with  $A_{kj} = 0$  if  $k > j$ . Then

$$\begin{aligned} \bar{A}_{11}A_{11} &= \sum_{k=1}^n \bar{A}_{k1}A_{k1} = \sum_{k=1}^n A_{1k}^*A_{k1} = (A^*A)_{11} = \\ &= (AA^*)_{11} = \sum_{k=1}^n A_{1k}A_{k1}^* = \sum_{k=1}^n A_{1k}\bar{A}_{1k}. \end{aligned}$$

It now follows that  $A_{12} = A_{13} = \dots = A_{1n} = 0$ . Consider the  $(2, 2)$  entry.

$$(AA^*)_{22} = \sum_{k=1}^n A_{2k}(A^*)_{k2} = \sum_{k=2}^n A_{2k}\bar{A}_{2k}.$$

This must also equal

$$(A^*A)_{22} = \sum_k (A^*)_{2k}A_{k2} = \sum_{k=1}^2 (A^*)_{2k}A_{k2} = \bar{A}_{22}A_{22}.$$

It follows that  $A_{23} = A_{24} = \dots = A_{2n}$ .

Proceed down the rows to show recursively that in fact  $A$  must be diagonal.

## 7. Tournament Matrices

The matrices of this problem are all  $n \times n$  with real entries.

- (a) Show that if the matrix  $A$  is skew-symmetric then  $I + A$  is nonsingular.
- (b) Show that for arbitrary matrices  $A$  and  $B$ ,  $\text{rk}(A + B) \leq \text{rk}(A) + \text{rk}(B)$ .
- (c) If  $A$  is arbitrary and  $J$  is the matrix of all 1's, then show that

$$\text{rk}(A - J) \geq \text{rk}(A) - \text{rk}(J).$$

- (d) If  $M$  is a  $(0, 1)$ -matrix with zeros on the main diagonal and with  $M_{ij} = 0$  if and only if  $M_{ji} = 1$ , show that  $\text{rk}(M) \geq n - 1$ . (Such a matrix is called a *tournament matrix*.)

**Solution:** Suppose  $A^T = -A$  and that  $X$  is a column vector for which  $(I + A)X = 0$ . Then  $AX = -X$  implies that  $X = (-A)X = A^T X$ , so  $X^T A = X^T$ . Then  $X^T X = (X^T A)X = X^T(AX) = X^T(-X) = -X^T X$ , which implies that  $X^T X = 0$ , and hence  $X = 0$ . So 0 is not an eigenvalue of  $I + A$ . For the second part, observe that the union of a maximal independent set of rows of  $A$  with a maximal independent set of rows of  $B$  will certainly span the row space of  $A + B$ . For the third part, apply the second part to the matrix  $A = (A - J) + J$ . For the last part, let  $M$  be a tournament matrix of order  $n$ . Then  $M + M^T = J - I$ , i.e.,  $J = I + M + M^T$ . Clearly  $M - M^T$  is skew-symmetric, so  $A = I + M - M^T$  is nonsingular by the first part. Hence  $\text{rk}(A) = n$ . Then  $\text{rk}(A - J) \geq \text{rk}(A) - \text{rk}(J) = n - 1$ . But  $A - J = -2M^T$ , so  $\text{rk}(M) = \text{rk}(M^T) = \text{rk}(A - J) \geq n - 1$ .

8. Given an  $m \times n$  matrix  $A$ , the *pseudoinverse* of  $A$ , denoted  $A^+$ , can be defined as the matrix such that for all  $b \in \mathcal{C}^m$ ,  $x^+ := A^+b$  is the least squares solution to the equation  $Ax = b$  that has the smallest norm.
- Using the above definition, explain why  $AA^+$  and  $A^+A$  must be projection matrices (and are therefore Hermitian). Onto what fundamental subspaces do these matrices project?
  - Prove that  $AA^+A = A$  and  $A^+AA^+ = A^+$ . (Note: these two properties, together with the Hermitian properties in part (a) uniquely determine the pseudoinverse).
  - If  $\Sigma$  is a real diagonal matrix, what is  $\Sigma^+$ ?
  - Give an explicit formula for  $A^+$  in terms of the singular value decomposition  $A = V\Sigma W^*$ . Justify your answer.

**Solution:**

- For  $x^+$  to be a least squares solution to  $Ax = b$ ,  $Ax^+$  must be the orthogonal projection of  $b$  onto the column space of  $A$ . Let  $p(b)$  be this projection. Then  $AA^+b = Ax^+ = p(b)$ . It follows that  $AA^+$  is the projection matrix onto the column space of  $A$ .  
Since  $x^+$  is the *least norm* solution to  $Ax = p(b)$ , it must lie in the row space of  $A$ .  
Consider any  $y \in \mathcal{C}^n$ . Let  $b = Ay$  and  $x^+ = A^+b = A^+Ay$ . Since  $b$  is in the column space of  $A$ ,  $p(b) = b$ . It follows that  $Ax^+ = p(b) = Ay$ , so  $A(x^+ - y) = 0$ . Thus  $x^+ = A^+Ay$  is the orthogonal projection onto the row space of  $A$ .
- Observe that  $A^+b = A^+p(b)$ . Thus, for any  $b$ ,  $A^+AA^+b = A^+Ax^+ = A^+p(b) = A^+b$ . Since this is true for all  $b$ ,  $A^+AA^+ = A^+$ . Similarly, for any  $y$ ,  $AA^+Ay = Ax^+ = Ay$ . Thus,  $AA^+A = A$ .
- $\Sigma^+$  is the diagonal matrix with entries

$$\Sigma_{ii}^+ = \begin{cases} 1/\Sigma_{ii}, & \text{if } \Sigma_{ii} \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to verify that  $\Sigma\Sigma^+ = \Sigma^+\Sigma$  is diagonal (and hence Hermitian);  $\Sigma\Sigma^+\Sigma = \Sigma$ , and  $\Sigma^+\Sigma\Sigma^+ = \Sigma^+$ . Thus  $\Sigma^+$  is the pseudoinverse.

(d)  $A^+ = W\Sigma^+V^*$ . To prove that this is the pseudoinverse, check each of the properties. Let  $D := \Sigma\Sigma^+$ , and observe that  $D$  is diagonal with  $D_{ii} = 0$  if  $\Sigma_{ii} = 0$ , and  $D_{ii} = 1$  otherwise. Then

- $AA^+ = V\Sigma W^*W\Sigma^+V^* = V\Sigma\Sigma^+V^* = VDV^*$ , which is clearly Hermitian.
- Similarly,  $A^+A = W\Sigma^+V^*V\Sigma W^* = WDW^*$ , which is Hermitian.
- $AA^+A = V\Sigma W^*W\Sigma^+V^*V\Sigma W^* = VD\Sigma W^* = V\Sigma W^* = A$ .
- $A^+AA^+ = W\Sigma^+V^*V\Sigma W^*V\Sigma^+V^* = WD\Sigma^+W^* = W\Sigma^+W^* = A^+$ .