University of Colorado at Denver — Mathematics Department

Applied Linear Algebra Preliminary Exam with Solutions

18 January 2008, 10:00 am – 2:00 pm

Name: ____

The proctor will let you read the following conditions before the exam begins, and you will have time for questions. Once the exam begins, you will have 4 hours to do your best. This is a closed book exam. Please put your name on each sheet of paper that you turn in.

Exam conditions:

- Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write legibly using a dark pencil or pen.
- Notation: $\mathcal{C}$ denotes the field of complex numbers , $\mathcal{R}$ denotes the field of real numbers, and F denotes a field which may be either $\mathcal{C}$ or $\mathcal{R}$. $\mathcal{C}^n$ and $\mathcal{R}^n$ denote the vector spaces of n-tuples of complex and real scalars, respectively. T^* is the adjoint of the operator T and $\bar{\lambda}$ is the complex conjugate of the scalar λ. v^T and A^T denote vector and matrix transposes, respectively.
- Ask the proctor if you have any questions.

Good luck!

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________	8.	________
		Total	________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

On this exam you may assume that V is a finite-dimensional vector space over F, say $\dim(V) = n$.

1. Let $T \in \mathcal{L}(V)$. Prove or disprove each of the following:

   (a) (10 points) $V = \text{null}(T) \oplus \text{range}(T)$.

   (b) (10 points) There exists a subspace U of V such that $U \cap \text{null}(T) = \{\mathbf{0}\}$ and $\text{range}(T) = \{T(u) : u \in U\}$.

   Solution: For part (a) define $T : \mathcal{R}^2 \to \mathcal{R}^2 : (x, y) \mapsto (y, 0)$. Then $T \in \mathcal{L}(\mathcal{R}^2)$ and $\{\mathbf{0}\} \neq \text{null}(T) = \text{range}(T) \neq \mathcal{R}^2$.

   For part (b), let $\mathcal{B}$ be a basis of null(T) and complete it to a basis $\mathcal{B} \cup \mathcal{B}'$ of all of V. Then if $U = \text{span}(\mathcal{B}')$, the space U satisfies the condition of the problem.

2. Let $V = \mathcal{R}^5$ and let $T \in \mathcal{L}(V)$ be defined by $T(a, b, c, d, e) = (2a, 2b, 2c + d, a + 2d, b + 2e)$.

   (a) (8 points) Find the characteristic and minimal polynomial of T.

   (b) (8 points) Determine a basis of F^5 consisting of eigenvectors and generalized eigenvectors of T.

   (c) (4 points) Find the Jordan form of T (or of $M(T)$) with respect to your basis.

   Solution: Let $\mathcal{S} = (\mathbf{e}_1, \ldots, \mathbf{e}_5)$ be the standard ordered basis of $\mathcal{R}^5$. Then

$$[T]_{\mathcal{S}} = A = \begin{pmatrix} 2 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

   The characteristic polynomial $f(x)$ of T is given by

$$f(x) = |xI - A| = \begin{vmatrix} x-2 & 0 & 0 & -1 & 0 \\ 0 & x-2 & 0 & 0 & -1 \\ 0 & 0 & x-2 & 0 & 0 \\ 0 & 0 & -1 & x-2 & 0 \\ 0 & 0 & 0 & 0 & x-2 \end{vmatrix} = (x-2)^5,$$

   where we use the fact hat the matrix is block upper triangular and both blocks are triangular. So we know that the characteristic polynomial of T is $(x - 2)^5$. This means that the only eigenvalue is 2 and there is a basis of $\mathcal{R}^5$ consisting of generalized eigenvectors belonging to $\lambda = 2$.

   We compute:

$$
\begin{aligned}
2I - A &= \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \\
(2I - A)^2 &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \\
(2I - A)^3 = \mathbf{0}.
\end{aligned}
$$

By inspection of these matrices we see that:

$$2I - T : \mathbf{e}_3 \mapsto -\mathbf{e}_4 \mapsto \mathbf{e}_1 \mapsto \mathbf{0}; \quad \mathbf{e}_5 \mapsto -\mathbf{e}_2 \mapsto \mathbf{0}.$$

So we form a new *ordered* basis

$$\mathcal{B} = (\mathbf{e}_3,\ \mathbf{e}_4,\ \mathbf{e}_1,\ \mathbf{e}_5,\ \mathbf{e}_2).$$

Then

$$[2I - T]_{\mathcal{B}} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{so that} \quad [T]_{\mathcal{B}} = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

This last matrix is in Jordan form, and we can see that the minimal polynomial of T is $p(x) = (x - 2)^3$.

3. Let A be an $m \times n$ complex matrix and let B be an $n \times m$ complex matrix. Also, for any positive integer p, I_p denotes the $p \times p$ identity matrix. So AB is $m \times m$, while BA is $n \times n$.

   (i) (5 points) Show that

$$\begin{pmatrix} AB & \mathbf{0} \\ B & \mathbf{0} \end{pmatrix} \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} I_m & A \\ \mathbf{0} & I_n \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ B & BA \end{pmatrix}.$$

(ii) (5 points) Show that $\begin{pmatrix} AB & \mathbf{0} \\ B & \mathbf{0} \end{pmatrix}$ and $\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ B & BA \end{pmatrix}$ are similar matrices, and hence have the same characteristic polynomials. (Here we let $\mathbf{0}$ denote the zero matrix of the uniquely determined appropriate size.)

(iii) (5 points) Show that AB and BA have the same characteristic polynomials up to a factor of a power of x. If $m = n$ they have exactly the same characteristic polynomials.

(iv) (5 points) Show that the complex number $\lambda \neq 1$ is an eigenvalue of $I_m - AB$ if and only if λ is an eigenvalue of $I_n - BA$.

Solution:

Part (i). Using block multiplication we see that both products are equal to $\begin{pmatrix} AB & ABA \\ B & BA \end{pmatrix}$.

Part(ii) The matrix $\begin{pmatrix} I_m & A \\ \mathbf{0} & I_n \end{pmatrix}$ is invertible with inverse

$$\begin{pmatrix} I_m & A \\ \mathbf{0} & I_n \end{pmatrix}^{-1} = \begin{pmatrix} I_m & -A \\ \mathbf{0} & I_n \end{pmatrix}.$$

Hence it follows immediately from Part (i) that the two given matrices are similar.

Part (iii) The characteristic polynomial of $\begin{pmatrix} AB & \mathbf{0} \\ B & \mathbf{0} \end{pmatrix}$ is equal to the determinant

$$\begin{vmatrix} xI_m - AB & \mathbf{0} \\ -B & xI_n \end{vmatrix} = |xI_m - AB| \cdot x^n.$$

Similarly, the characteristic polynomial of $\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ B & BA \end{pmatrix}$ is $x^m\,|xI_n - BA|$. It follows that if $f(x)$ is the characteristic polynomial of AB and $g(x)$ is the characteristic polynomial of BA, then $x^n f(x) = x^m g(x)$. From this it is clear that if λ is nonzero, then λ has the same algebraic multiplicity as an eigenvalue of AB as it has for BA. What about $\lambda = 0$? Try $A = (1,1)$ and $B = (1,1)^T$. Then $AB = (2)$ is nonsingular, but $BA = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ has $\lambda = 0$ as an eigenvalue.

Part (iv). Let $\lambda \neq 1$. Then λ is an eigenvalue of $I_m - AB$ if and only if $1 - \lambda \neq 0$ is an eigenvalue of AB if and only if $1 - \lambda \neq 0$ is an eigenvalue of BA if and only if λ is an eigenvalue of $I_n - BA$. Hence $I_m - AB$ and $I_n - BA$ have exactly the same eigenvalues different from 1.

4. **Problem:**Suppose that $S, T \in \mathcal{L}(V)$.

(a) (10 points) Prove that ST and TS have the same eigenvalues.

(b) (10 points) Suppose that T has $\dim(V)$ distinct eigenvalues and that S has the same eigenvectors as T (though not necessarily with the same eigenvalues). Prove that $ST = TS$.

Solution: The most elegant solution of part (a) of this problem is merely to quote part (iii) of Prob. 3 for the nonzero eigenvalues. For $\lambda = 0$ we note that ST has zero as an eigenvalue if and only if it is not invertible if and only if one of S, T is not invertible, if and only if TS is not invertible if and only if TS has zero as an eigenvalue.

Here is another approach. Suppose that $ST(v) = \lambda v$ with $v \neq \mathbf{0}$. If $T(v) = \mathbf{0}$, then both ST and TS have 0 as an eigenvalue. If $T(v) \neq \mathbf{0}$, then

$TS(T(v)) = T(\lambda v) = \lambda T(v)$, so that λ is also an eigenvalue of TS. Since every eigenvalue of ST is an eigenvalue of TS, interchanging the roles of S and T shows that ST and TS have the same eigenvalues.

For part (b), the hypothesis on T says that if $\dim(V) = n$, there is a basis $\mathcal{B} = (v_1, \ldots, v_n)$ consisting of eigenvectors associated with the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively. In particular we note that a vector v is an eigenvector of T if and only if it is a scalar multiple of one of the special basis vectors. Since S has the exactly the same eigenvectors as T, no eigenspace of S can have dimension greater than 1. Hence there must be distinct eigenvalues $\mu_1, \ldots, \mu_n$ of S such that $S(v_j) = \mu_j v_j$. Then

$$ST(v_j) = S(\lambda_j v_j) = \lambda_j S(v_j) = \lambda_j \mu_j v_j = \mu_j(\lambda_j v_j) = \mu_j T(v_j) = T(\mu_j v_j) = TS(v_j).$$

Since ST and TS agree on a basis of V, it must be that $ST = TS$.

5. **Problem** In this problem $M_n(\mathcal{R})$ denotes the set of all $n \times n$ real matrices, and $\mathcal{R}^n$ denotes the usual real inner product space of all column vectors with n real entries and inner product $(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T \mathbf{y} = \mathbf{y}^T \mathbf{x} = (\mathbf{y}, \mathbf{x})$.

Define the following subsets of $M_N(\mathcal{R})$:

$$\begin{aligned}
\Pi_n &= \{A \in M_n(\mathcal{R}) : (A\mathbf{x}, \mathbf{x}) > 0 \text{ for all } \mathbf{0} \neq \mathbf{x} \in \mathcal{R}^n\} \\
\mathcal{S}_n &= \{A \in M_n(\mathcal{R}) : A^T = A\} \\
\Sigma_n &= \Pi_n \cap \mathcal{S}_n \\
K_n &= \{A \in M_n(\mathcal{R}) : A^T = -A\}
\end{aligned}$$

For this problem we say that $A \in M_n(\mathcal{R})$ is *positive definite* if and only if $A \in \Pi_n$. A is *symmetric* if and only if it belongs to $\mathcal{S}_n$. It is *skew-symmetric* if and only if it belongs to K_n.

Note: It is sometimes the case that a real matrix is said to be positive definite if and only if it belongs to $\mathcal{S}_n$ also, i.e. $A \in \Sigma_n$. Remember that we do not do that here. Moreover, Axler calls a real matrix *positive* if it is both symmetric and *positive semidefinite*, i.e., $(Au, u) \geq 0$ for all $u \in V$. However, our definition here is a very common one.

Let $A \in M_n(\mathcal{R})$.

(i) (4 points) Prove that there are *unique* matrices $B \in \mathcal{S}_n$ and $C \in K_n$ for which $A = B + C$. Here B is called the *symmetric part of* A and C is called the *skew-symmetric part of* A.

(ii) (4 points) Show that A is positive definite if and only if the symmetric part of A is positive definite.

(iii) (4 points) Let A be symmetric. Show that A is positive definite if and only if all eigenvalues of A are positive. If this is the case, then $\det(A) > 0$.

(iv) (4 points) Let $\emptyset \neq S \subseteq \{1, 2, \ldots, n\}$. Let A_S denote the submatrix of A formed by using the rows and columns of A indexed by the elements of S. (So in particular if $S = \{k\}$, then A_S is the 1×1 matrix whose entry is the diagonal entry A_{kk} of A.) Prove that if A is positive definite (but not necessarily symmetric), then A_S is positive definite. Hence conclude that each diagonal entry of A is positive.

(v) (4 points) Let $A \in \Sigma_n$ and for $1 \leq i \leq n$ let A_i denote the principal submatrix of A defined by using the first i rows and first i columns of A. Prove that $\det(A_i) > 0$ for each $i = 1, 2, \ldots, n$. (Note: The converse is also true, but the proof is a bit more difficult.)

Solution:

Part (i).

$$A = \frac{A + A^T}{2} + \frac{A - A^T}{2} = B + C,$$

where $B = \frac{A+A^T}{2}$ is symmetric and $C = \frac{A-A^T}{2} = B + C$ is skew-symmetric. Suppose that $A = B + C = B' + C'$ where B and B' are symmetric and C and C' are skew-symmetric. Then $B - B' = C' - C$, where $B - B'$ is symmetric and $C' - C$ is skew-symmetric. It follows that $B - B' = \mathbf{0}$, forcing B and C to be unique.

Part (ii). Using the decomposition above, we have $(Av, v) = ((B + C)v, v) = (Bv, v) + (Cv, v)$. But since C is skew-symmetric, we have

$$(Cv, v) = v^T C^T v = -v^T Cv = -(v, Cv) = -(Cv, v), \text{ implying } (Cv, v) = 0.$$

Hence $(Av, v) = (Bv, v)$ for all $v \in \mathcal{R}^n$. Hence if $(Av, v) > 0$ for all nonzero v, clearly $(Cv, v) > 0$ for all nonzero v.

Part (iii). Let A be real and symmetric. So the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A are all real and there is a (real) orthogonal matrix P (i.e., $P^T = P^{-1}$) such that $P^T AP = D$ is the diagonal matrix $\text{diag}(\lambda_1, \ldots, \lambda_n)$.

First suppose that each $\lambda_i > 0$, $1 \leq i \leq n$. Let $\mathbf{x}$ be an arbitrary *non-zero* vector in $\mathcal{R}^n$ and put $\mathbf{y} = P^T\mathbf{x}$, so $\mathbf{y} \neq \mathbf{0}$ and $\mathbf{x} = P\mathbf{y}$. Then

$$(A\mathbf{x}, \mathbf{x}) = \mathbf{x}^T A\mathbf{x} = \mathbf{y}^T P^T AP\mathbf{y} = \mathbf{y}^T D\mathbf{y} = \sum_{i=1}^{n} \lambda_i y_i^2 > 0,$$

since each $\lambda_i > 0$ and at least one $y_i^2 > 0$. Hence A is positive definite.

Conversely, suppose that A is positive definite. Since A is real symmetric there is a basis $(v_1, v_2, \ldots, v_n)$ of $\mathcal{R}^n$ consisting of eigenvectors of A: $Av_i = \lambda_i v_i$. Then by hypothesis

$$0 < (Av_i, v_i) = \lambda_i(v_i, v_i), \text{ hence } \lambda_i > 0 \text{ since } (v_i, v_i) > 0.$$

Since the determinant of A is just the product of its eigenvalues, if they are all positive, so is $\det(A)$.

Part (iv) Start with an arbitrary non-empty subset S of $\{1, 2, \ldots, n\}$ and let $A \in \Pi_n$. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in \mathcal{R}^n$ be such that $x_i = 0$ if $i \notin S$ but $\mathbf{x} \neq \mathbf{0}$. Let $\mathbf{x}_S$ be the vector obtained from $\mathbf{x}$ by suppressing the (zero entries) x_j where $j \notin S$. Then $0 < \mathbf{x}^T A\mathbf{x} = \mathbf{x}_S^T A_S \mathbf{x}_S$. Since $\mathbf{x}_S$ is arbitrary (non-zero), we see that A_S is positive definite. In particular, if $S = \{k\}$, the submatrix A_S being positive definite implies that $A_{kk} > 0$.

Part (v). We are assuming that A is symmetric and positive definite. Clearly each A_i is symmetric and by Part (iv) it is positive definite, i.e., $A_i \in \Sigma_i$. Hence by Part (iii) all the eigenvalues of A_i are positive, forcing $\det(A_i) > 0$.

6. **Problem:**

Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathcal{C}[x]$. Let T be the linear operator on $\mathcal{C}^n$ whose matrix with respect to the standard basis $\mathcal{S} = (e_1, e_2, \ldots, e_n)$ is the *companion matrix of* $f(x)$, i.e., $C(f(x))$ is defined by:

$$C(f(x)) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & -a_{n-2} \\ 0 & 0 & \cdots & 0 & 1 & -a_{n-1} \end{pmatrix}.$$

(i) (10 points) Show that $f(x)$ is both the characteristic and minimal polynomial of the matrix $C(f(x))$.

(ii) (10 points) An *algebraic integer* is a complex number λ that is a root of a monic polynomial (leading coefficient equals 1) with integer coefficients. Show that a complex number λ is an algebraic integer if and only if it is an eigenvalue of a square matrix with integer entries.

Solution: Part (i).

First we establish that $f(x) = det(xI_n - C(f(x)))$. This result is clear if $n = 1$ and we proceed by induction. Suppose that $n > 1$ and compute the determinant by cofactor expansion along the first row, applying the induction hypothesis to the first summand.

$$det(xI_n - C(f(x))) = det \begin{pmatrix} x & 0 & \cdots & 0 & 0 & a_0 \\ -1 & x & \cdots & 0 & 0 & a_1 \\ 0 & -1 & \cdots & 0 & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & x & a_n - 2 \\ 0 & 0 & \cdots & 0 & -1 & x + a_{n-1} \end{pmatrix}.$$

$$
= x \, det \begin{pmatrix} x & \cdots & 0 & 0 & a_1 \\ -1 & \cdots & 0 & 0 & a_2 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & -1 & x & a_{n-2} \\ 0 & \cdots & 0 & -1 & x + a_{n-1} \end{pmatrix} +
$$

$$
+a_0(-1)^{n+1} det \begin{pmatrix} -1 & x & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & x \\ 0 & 0 & \cdots & 0 & -1 \end{pmatrix}
$$

$$
= x(x^{n-1} + a_{n-1}x^{n-2} + \cdots + a_1) + a_0(-1)^{n+1}(-1)^{n-1}
$$

$$
= x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = f(x).
$$

This shows that $f(x)$ is the characteristic polynomial of $C(f(x))$. Then $Te_1 = e_2$, $T^2e_1 = Te_2 = e_3, \ldots, T^j e_1 = T(e_j) = e_{j+1}$ for $1 \le j \le n-1$, and $Te_n = -a_0e_1 - a_1e_2 - \cdots - a_{n-1}e_n$, so

$$
(T^n + a_{n-1}T^{n-1} + \cdots + a_1T + a + 0I)e_1 = \bar{0}.
$$

Also

$$
\begin{aligned}
(T^n + \cdots + a_1T + a_0I)e_{j+1} &= (T^n + \cdots + a_1T + a_0I)T^j e_1 \\
&= T^j(T^n + \cdots + a_1T + a_0I)e_1 = \bar{0}.
\end{aligned}
$$

It follows that $f(T)$ must be the zero operator. On the other hand, $(e_1, Te_1, \ldots, T^{n-1}e_1)$ is a linearly independent list, so that no nonzero polynomial in T with degree less than n can be the zero operator. Then since $f(x)$ is monic it must be that $f(x)$ is also the minimal polynomial for T and hence for $C(f(x))$. This completes the proof of part (i).

Part (ii).

Let λ be an algebraic integer, so there is a monic polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ with integer coefficients such that $f(\lambda) = 0$. Let A be the companion matrix of $f(x)$. Then $f(x)$ is the characteristic polynomial of A and $f(\lambda) = 0$, so λ is an eigenvalue of the square matrix A with integer entries.

Conversely, suppose that λ is an eigenvalue of a square matrix A with integer entries. Let $f(x)$ be the characteristic polynomial (or the minimal polynomial) of A. Then $f(x)$ is monic with integer coefficients and $f(\lambda) = 0$.

For the following two problems let $F = \mathcal{C}$ and let $V = \mathcal{C}^n$ be the usual complex inner product space.

7. (10 points each direction) Let $u, v \in V$. Prove that $\langle u, v\rangle = 0$ iff $\|u\| \leq \|u + av\|$ for all $a \in F$.

Solution: First suppose that $\langle u, v\rangle = 0$. By the Pythagorean theorem

$$\|u + av\|^2 = \|u\|^2 + \|av\|^2 \geq \|u\|^2.$$

Taking square roots gives $\|u\| \leq \|u+av\|$ for all $a \in F$. For the converse, assume that this condition holds. The we have

$$\begin{aligned} \|u\|^2 &\leq \|u + av\|^2 \\ &= \langle u + av, u + av\rangle \\ &= \langle u, u\rangle + \langle u, av\rangle + \langle av, u\rangle + \langle av, av\rangle \\ &= \|u\|^2 + \bar{a}\langle u, v\rangle + a\langle v, u\rangle + a\bar{a}\langle v, v\rangle. \\ &= \|u\|^2 + 2\text{Real}(\bar{a}\langle u, v\rangle) + |a|^2\|v\|^2 \text{ for all } a \in F. \end{aligned}$$

It follows that

$$-2\text{Real}(\bar{a}\langle u, v\rangle) \leq |a|^2\|v\|^2 \quad \text{for all } a \in F.$$

In particular put $a = -t\langle u, v\rangle$ for $t > 0$ to obtain

$$2t|\langle u, v\rangle|^2 \leq t^2|\langle u, v\rangle|^2\|v\|^2 \quad \text{for all } t > 0.$$

If $v = \mathbf{0}$, then $\langle u, v\rangle = 0$ as desired. Suppose $v \neq \mathbf{0}$. Then divide both sides by t, after which put $t = \frac{1}{\|b\|^2}$ to obtain

$$2|\langle u, v\rangle|^2 \leq |\langle u, v\rangle|^2.$$

This implies that $\langle u, v\rangle = 0$, completing the proof.

8. Let $R, S, T \in \mathcal{L}(V)$.

   (i) (10 points) Suppose that S is an isometry and R is a positive operator (Axler's definition) such that $T = SR$. Prove that $R = \sqrt{T^*T}$.

   (ii) (10 points) Let σ denote the smallest singular value of T, and let σ^* denote the largest singular value of T. Prove that $\sigma \leq \left\|\frac{T(v)}{\|v\|}\right\| \leq \sigma^*$ for every nonzero $v \in V$.

Solution Part (i) The hypothesis that S be an isometry means that $S^*S = I$. The hypothesis that R be a a positive operator (by Axler's definition) means that R is self-adjoint. Taking adjoints of both sides of $T = SR$ gives $T^* = R^*S^*$ so that $T^*T = R^*S^*SR = R^*R = R^2$. Then since R is a square root of T^*T and R is positive, we have that $R = \sqrt{T^*T}$.

Part (ii). Let $v \in V$. By the singular value decomposition there exist orthonormal bases $(u_1, \ldots, u_n)$ and $(w_1, \ldots, w_n)$ of V such that

$$T(v) = \sigma_1\langle v, u_1\rangle w_1 + \cdots + \sigma_n(\langle v, u_n\rangle w_n,$$

where $\sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_n$ are the singular values of T. Because $(u_1, \ldots, u_n)$ and $(w_1, \ldots, w_n)$ are both orthonormal bases of V, we have

$$\begin{aligned} \sigma_1^2\|v\|^2 &= \sigma_1^2(|\langle v, u_1\rangle|^2 + \cdots + |\langle v, u_n\rangle|^2) \\ &\leq \sigma_1^2|\langle v, u_1\rangle|^2 + \cdots + \sigma_n^2|\langle v, u_n\rangle|^2 = \\ = \|T(v)\|^2 &\leq \sigma_n^2(|\langle v, u_1\rangle|^2 + \cdots + |\langle v, u_n\rangle|^2) = \sigma_n^2\|v\|^2, \end{aligned}$$

giving both inequalities.