University of Colorado at Denver — Mathematics Department

Applied Linear Algebra Preliminary Exam

18 January 2008, 10:00 am – 2:00 pm

Name: ____

The proctor will let you read the following conditions before the exam begins, and you will have time for questions. Once the exam begins, you will have 4 hours to do your best. This is a closed book exam. Please put your name on each sheet of paper that you turn in.

Exam conditions:

- Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write legibly using a dark pencil or pen.
- Notation: $\mathcal{C}$ denotes the field of complex numbers , $\mathcal{R}$ denotes the field of real numbers, and F denotes a field which may be either $\mathcal{C}$ or $\mathcal{R}$. $\mathcal{C}^n$ and $\mathcal{R}^n$ denote the vector spaces of n-tuples of complex and real scalars, respectively. T^* is the adjoint of the operator T and $\bar{\lambda}$ is the complex conjugate of the scalar λ. v^T and A^T denote vector and matrix transposes, respectively.
- Ask the proctor if you have any questions.

Good luck!

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________	8.	________

Total ________

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On this exam you may assume that V is a finite-dimensional vector space over F, say $\dim(V) = n$.

1. Let $T \in \mathcal{L}(V)$. Prove or disprove each of the following:

   (a) (10 points) $V = \text{null}(T) \oplus \text{range}(T)$.

   (b) (10 points) There exists a subspace U of V such that $U \cap \text{null}(T) = \{\mathbf{0}\}$ and $\text{range}(T) = \{T(u) : u \in U\}$.

2. Let $V = \mathcal{R}^5$ and let $T \in \mathcal{L}(V)$ be defined by $T(a, b, c, d, e) = (2a, 2b, 2c + d, a + 2d, b + 2e)$.

   (a) (8 points) Find the characteristic and minimal polynomial of T.

   (b) (8 points) Determine a basis of F^5 consisting of eigenvectors and generalized eigenvectors of T.

   (c) (4 points) Find the Jordan form of T (or of $M(T)$) with respect to your basis.

3. Let A be an $m \times n$ complex matrix and let B be an $n \times m$ complex matrix. Also, for any positive integer p, I_p denotes the $p \times p$ identity matrix. So AB is $m \times m$, while BA is $n \times n$.

   (i) (5 points) Show that

$$\begin{pmatrix} AB & \mathbf{0} \\ B & \mathbf{0} \end{pmatrix} \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} I_m & A \\ \mathbf{0} & I_n \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ B & BA \end{pmatrix}.$$

   (ii) (5 points) Show that $\begin{pmatrix} AB & \mathbf{0} \\ B & \mathbf{0} \end{pmatrix}$ and $\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ B & BA \end{pmatrix}$ are similar matrices, and hence have the same characteristic polynomials. (Here we let $\mathbf{0}$ denote the zero matrix of the uniquely determined appropriate size.)

   (iii) (5 points) Show that AB and BA have the same characteristic polynomials up to a factor of a power of x. If $m = n$ they have exactly the same characteristic polynomials.

   (iv) (5 points) Show that the complex number $\lambda \neq 1$ is an eigenvalue of $I_m - AB$ if and only if λ is an eigenvalue of $I_n - BA$.

4. Suppose that $S, T \in \mathcal{L}(V)$.

   (i) (10 points) Prove that ST and TS have the same eigenvalues.

   (ii) (10 points) Suppose that T has $\dim(V)$ distinct eigenvalues and that S has the same eigenvectors as T (though not necessarily with the same eigenvalues). Prove that $ST = TS$.

5. In this problem $M_n(\mathcal{R})$ denotes the set of all $n \times n$ real matrices, and $\mathcal{R}^n$ denotes the usual real inner product space of all column vectors with n real entries and inner product $(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T\mathbf{y} = \mathbf{y}^T\mathbf{x} = (\mathbf{y}, \mathbf{x})$.

Define the following subsets of $M_N(\mathcal{R})$:

$$\begin{aligned} \Pi_n &= \{A \in M_n(\mathcal{R}) : (A\mathbf{x}, \mathbf{x}) > 0 \text{ for all } \mathbf{0} \neq \mathbf{x} \in \mathcal{R}^n\} \\ \mathcal{S}_n &= \{A \in M_n(\mathcal{R}) : A^T = A\} \\ \Sigma_n &= \Pi_n \cap \mathcal{S}_n \\ K_n &= \{A \in M_n(\mathcal{R}) : A^T = -A\} \end{aligned}$$

For this problem we say that $A \in M_n(\mathcal{R})$ is *positive definite* if and only if $A \in \Pi_n$. A is *symmetric* if and only if it belongs to $\mathcal{S}_n$. It is *skew-symmetric* if and only if it belongs to K_n.

Note: It is sometimes the case that a real matrix is said to be positive definite if and only if it belongs to $\mathcal{S}_n$ also, i.e. $A \in \Sigma_n$. Remember that we do not do that here. Moreover, Axler calls a real matrix *positive* if it is both symmetric and *positive semidefinite*, i.e., $(Au, u) \geq 0$ for all $u \in V$. However, our definition here is a very common one.

Let $A \in M_n(\mathcal{R})$.

(i) (4 points) Prove that there are *unique* matrices $B \in \mathcal{S}_n$ and $C \in K_n$ for which $A = B + C$. Here B is called the *symmetric part of* A and C is called the *skew-symmetric part of* A.

(ii) (4 points) Show that A is positive definite if and only if the symmetric part of A is positive definite.

(iii) (4 points) Let A be symmetric. Show that A is positive definite if and only if all eigenvalues of A are positive. If this is the case, then $\det(A) > 0$.

(iv) (4 points) Let $\emptyset \neq S \subseteq \{1, 2, \ldots, n\}$. Let A_S denote the submatrix of A formed by using the rows and columns of A indexed by the elements of S. (So in particular if $S = \{k\}$, then A_S is the 1×1 matrix whose entry is the diagonal entry A_{kk} of A.) Prove that if A is positive definite (but not necessarily symmetric), then A_S is positive definite. Hence conclude that each diagonal entry of A is positive.

(v) (4 points) Let $A \in \Sigma_n$ and for $1 \leq i \leq n$ let A_i denote the principal submatrix of A defined by using the first i rows and first i columns of A. Prove that $\det(A_i) > 0$ for each $i = 1, 2, \ldots, n$. (Note: The converse is also true, but the proof is a bit more difficult.)

6. **Problem:**

Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathcal{C}[x]$. Let T be the linear operator on $\mathcal{C}^n$ whose matrix with respect to the standard basis $\mathcal{S} = (e_1, e_2, \ldots, e_n)$ is the *companion matrix of* $f(x)$, i.e., $C(f(x))$ is defined by:

$$C(f(x)) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & -a_{n-2} \\ 0 & 0 & \cdots & 0 & 1 & -a_{n-1} \end{pmatrix}.$$

(i) (10 points) Show that $f(x)$ is both the characteristic and minimal polynomial of the matrix $C(f(x))$.

(ii) (10 points) An *algebraic integer* is a complex number λ that is a root of a monic polynomial (leading coefficient equals 1) with integer coefficients. Show that a complex number λ is an algebraic integer if and only if it is an eigenvalue of a square matrix with integer entries.

For the following two problems let $F = \mathcal{C}$ and let $V = \mathcal{C}^n$ be the usual complex inner product space.

7. (10 points each direction) Let $u, v \in V$. Prove that $\langle u, v\rangle = 0$ iff $\|u\| \leq \|u + av\|$ for all $a \in F$.

8. Let $R, S, T \in \mathcal{L}(V)$.

(i) (10 points) Suppose that S is an isometry and R is a positive operator (Axler's definition) such that $T = SR$. Prove that $R = \sqrt{T^*T}$.

(ii) (10 points) Let σ denote the smallest singular value of T, and let σ^* denote the largest singular value of T. Prove that $\sigma \leq \left\| \frac{T(v)}{\|v\|} \right\| \leq \sigma^*$ for every nonzero $v \in V$.