Maximal and minimal polyiamonds

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Abstract

The min perimeter of a polyiamond with area n is whichever of $\lceil \sqrt{6n} \rceil + \{0, 1\}$ has the same parity as n, and the max area of a polyiamond with perimeter p is $\text{round}\left(p^2/6\right) - \{0, 1\}$, where we choose 0 iff $p \equiv 0 \pmod 6$.

These results are related as follows: we use max polyiamonds (polyiamonds with given perimeter and max area) to get a lower bound on min perimeter, and we use this result to construct min polyiamonds (polyiamonds with given area and min perimeter).

Key words: polyiamond, perimeter, area

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1 Polyiamond

Consider a triangular grid, also called a hexagonal grid. A *polyiamond* is a connected set of triangles on a triangular grid. See Figure 1. An *n-polyiamond* is a polyiamond with n triangles. A *quasipolyiamond* is defined similarly, except that the set of triangles need not be connected.

We use "natural units" to define the perimeter and area of a polyiamond. The *perimeter* of a polyiamond is the number of (boundary) edges. The *area* of a polyiamond is the number of triangles.

A polyiamond is *minimal* (or *optimal*) iff it has min perimeter with respect to all polyiamonds with the same area. (The use of the term "optimal" comes from the domain decomposition problem, discussed later.) A polyiamond is

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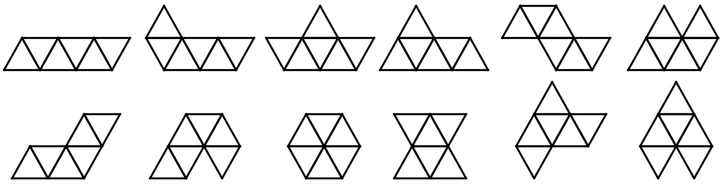

Fig. 1. The 12 polyiamonds with 6 triangles. All have perimeter 8 except for the regular hexagon, which has perimeter 6.

maximal iff it has max area with respect to all polyiamonds with the same perimeter.

Note that there are 2 other possibilities: we could maximize the perimeter subject to a fixed area, or minimize the area subject to a fixed perimeter. But these problems are trivial and have a common solution: a polyiamond shaped like a stick. An n-polyiamond shaped like a stick has perimeter $2n+2$, as opposed to a min n-polyiamond, whose perimeter is whichever of $\lceil\sqrt{6n}\rceil+\{0,1\}$ has the same parity as n (see the Min polyiamond formula (Theorem 7)).

2 Summary of results

- **Max polyiamond formula**: The max area of a polyiamond with perimeter p is

$$\operatorname{round}\left(\frac{p^2}{6}\right)-\begin{cases}0 & p\equiv 0 \pmod 6\\ 1 & \text{else}\end{cases}$$

- **Max/min polyiamond spiral algorithm**: This algorithm constructs a polyiamond with given perimeter and max area, and a polyiamond with min perimeter and given area.
- **Min polyiamond formula**: The min perimeter of an n-polyiamond is whichever of $\lceil\sqrt{6n}\rceil+\{0,1\}$ has the same parity as n.

These results are related as follows: we use max polyiamonds to get a lower bound on min perimeter, and then we use this result to construct min polyiamonds.

Motivation for this paper came from the domain decomposition problem with squares, which is a whole topic by itself ([1], [4], [7], [8]). The many approaches to this problem include branch-and-bound, genetic algorithms, knapsack algorithms, and stripe algorithms.

Below, we briefly discuss some known results in the domain decomposition problem with squares. These results will motivate the polyiamond results in this paper.

3 Polyomino

A *polyomino* is an edge-connected set of squares on a square grid. See Figure 2. An *n-polyomino* is a polyomino with n squares. "Polyomino" is a generalization of "domino". A *quasipolyomino* is defined similarly, except that the squares need not be edge-connected.

We use "natural units" to define the perimeter and area of a polyomino. The *perimeter* of a polyomino is the number of (boundary) edges. The *area* of a polyomino is the number of squares.

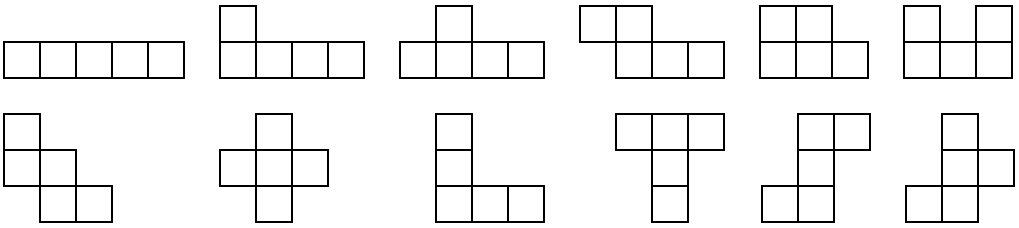

Fig. 2. The 12 polyominos with 5 squares. All have perimeter 12 except for one shaped like a 1×1 square joined to a 2×2 square, which has perimeter 10.

4 Domain decomposition with squares

We consider the following problem.

Problem 1 (Domain decomposition with squares) *Tile a given set of squares on a square grid by n-quasipolyominos, where n divides the size of the set. What is the min total perimeter of the quasipolyominos in such a tiling?*

Besides squares, the only regular polygons that tile the plane are equilateral triangles and regular hexagons.

Domain decomposition with squares has motivation from parallel computation; think of the following analogy:

square	job that needs to communicate with adjacent jobs
n-quasipolyomino	n jobs assigned to a processor
quasipolyomino edge	expensive communication between jobs in different processors

In a decomposition, unless the domain is disconnected or has an unusual shape, all the subdomains should be connected to minimize the total perimeter. If each subdomain has min perimeter, then the problem is solved. In general,

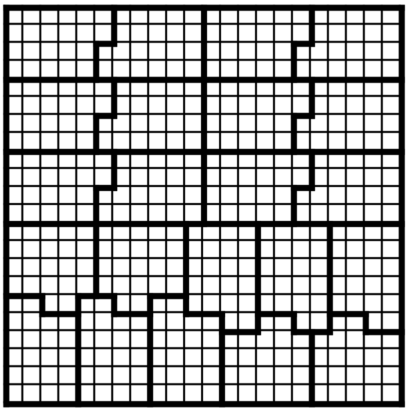

Fig. 3. Optimal decomposition of a 22×22 domain into subdomains of size 22. Each subdomain has the min perimeter of 20 (see the Min polyomino formula (Theorem 2)). The total perimeter is $22 \times 20 = 440$.

not all subdomains can have min perimeter, and the goal is to get as close as possible to the "all-min-perimeter" situation. This is usually not an easy task because the number of decompositions can be enormous, making an exhaustive search impractical.

Yackel-Meyer-Christou [8] discovered the following formula.

Theorem 2 (Min polyomino formula) *The min perimeter of an n-polyomino is* $2\left\lceil\sqrt{4n}\,\right\rceil$.

The perimeter of a polyomino is related to its numbers of "subslices". A *slice* is a row or column containing squares; the squares need not be connected. A *subslice* is a maximal connected set of squares in a slice. A *slice-gap* is an absence of squares between subslices in a row or column. A slice is *convex* iff the set of squares in the slice is convex; the slice has no gaps. A polyomino is *slice-convex* iff every slice is convex.

Theorem 3 (Polyomino subslices theorem) *The perimeter of a polyomino is 2 times the number of subslices.*

PROOF. Every subslice contributes 2 boundary edges. See Figure 4.

The following theorem follows immediately.

Theorem 4 (Polyomino slices theorem) *The perimeter of a slice-convex polyomino is 2 times the number of slices.*

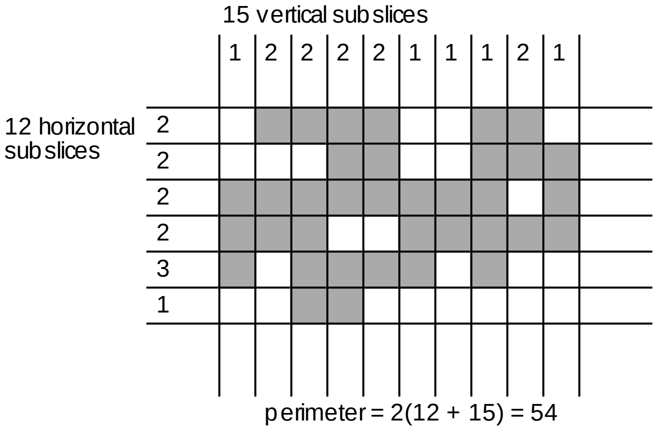

Fig. 4. The perimeter of a polyomino is 2 times the number of subslices.

5 Min polyiamond formula

Of all the polyiamond results summarized previously, the min polyiamond formula is the most important, because we use it all the time in domain decomposition. It enables us to get lower bounds for domain decomposition problems, and we measure the quality of an algorithm by seeing if it can produce values that are close to the lower bound.

The proof of the min polyiamond formula requires only 2 results, a parity result which we prove here, and a bounds result which we cite now and prove later.

Theorem 5 (Polyiamond perimeter parity) *The perimeter of an n-polyiamond has the same parity as n.*

PROOF. This result is a special case of the following result.

Theorem 6 (Quasipolyiamond perimeter parity) *The perimeter of an n-quasipolyiamond has the same parity as n.*

PROOF. Use induction. The base case is $n = 0$. There is 1 n-quasipolyiamond, and its perimeter is 0. Assume the statement is true for some $n \geq 0$. Note an $(n + 1)$-quasipolyiamond can be constructed by adding 1 triangle to an n-quasipolyiamond. This new triangle creates some edges and destroys some edges; the perimeter change is the net change in edges. Define the *degree* of a triangle as the number of triangles adjacent to it. Consider the degree of the new triangle.

degree	edges created	edges destroyed	perimeter change
0	3	0	+3
1	2	1	+1
2	1	2	−1
3	0	3	−3

Table 1
Change in quasipolyiamond perimeter from a new triangle.

By the table above, the perimeter change is always odd. So the new perimeter and the old perimeter have different parity.

By the induction hypothesis, the old perimeter has the same parity as n. So the new perimeter has parity different from n. So the new perimeter has the same parity as $n+1$.

Theorem 7 (Min polyiamond formula) *The min perimeter of an n-polyiamond is whichever of $\left\lceil \sqrt{6n} \right\rceil + \{0,1\}$ has the same parity as n.*

PROOF. Let minPerimeter(n) be the min perimeter of an n-polyiamond. By the Min polyiamond bounds (Theorem 18),

$$\left\lceil \sqrt{6n} \right\rceil \leq \text{minPerimeter}(n) \leq \left\lceil \sqrt{6n} \right\rceil + 1.$$

The bounds differ by 1, so the bounds have different parity. In particular, one of the bounds has the same parity as n. By the polyiamond perimeter parity theorem, minPerimeter(n) has the same parity as n. So minPerimeter(n) equals the bound with the same parity as n.

We motivate the appearance of $\sqrt{6n}$ in the Min polyiamond formula (Theorem 7). Consider shaping an n-polyiamond into a regular hexagon (see Figure 5). For the moment, we stop using natural units for area, so that the area of the polyiamond is not n, but $n\sqrt{3}/4$; each triangle has side length 1 and area $\sqrt{3}/4$. It is easy to verify that the perimeter of the hexagon is $\sqrt{6n}$.

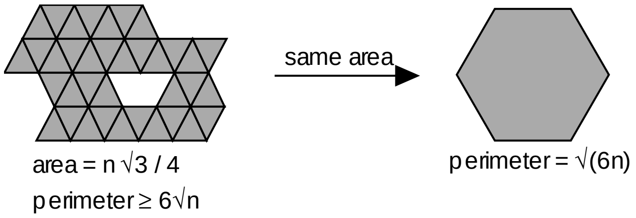

Fig. 5. Relation between polyiamond and regular hexagon of area n.

6 Domain decomposition with triangles

We consider the following problem.

Problem 8 (Domain decomposition with triangles) *Tile a given set of triangles on a triangular grid by n-quasipolyiamonds, where n divides the size of the set. What is the min total perimeter of the quasipolyiamonds in such a tiling?*

We generalize the slice approach used with polyominos. With polyominos, we have 2 kinds of slices: horizontal and vertical. But with polyiamonds, we have 3 kinds of slices: horizontal, antidiagonal, and diagonal ("HAD"). For brevity, we say that a polyiamond has *HAD slices (or dimensions)* (h, a, d) iff it has h horizontal slices, a antidiagonal slices, and d diagonal slices.

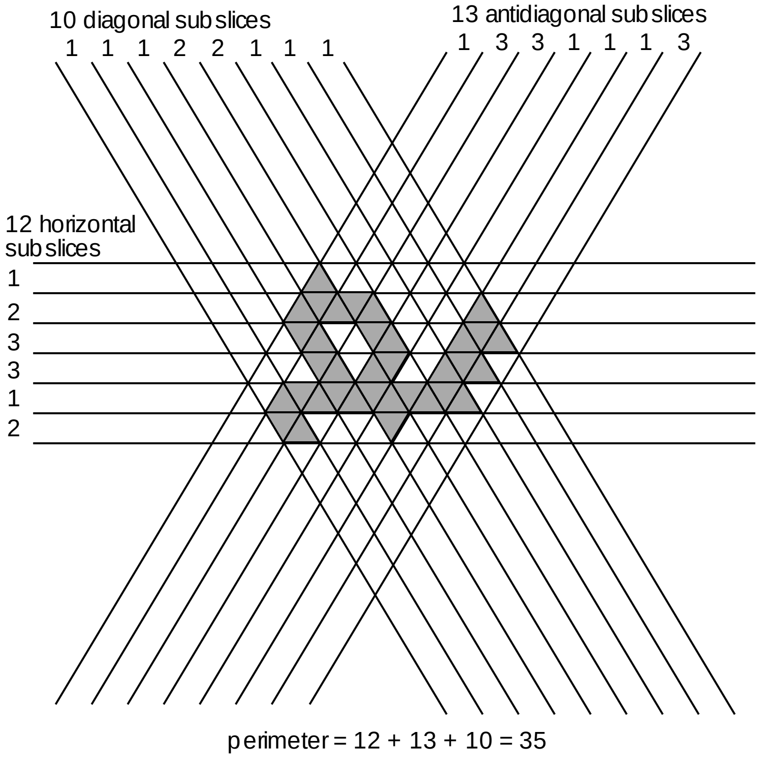

Fig. 6. The perimeter of a polyiamond is the number of subslices.

Theorem 9 (Polyiamond subslices theorem) *The perimeter of a polyiamond is the number of subslices.*

PROOF. See Figure 6. Note each of the 2 ends of every subslice is a boundary edge. Also, every boundary edge is the end of exactly 2 subslices. So the number of subslices equals the number of boundary edges, which equals the perimeter.

For a slice-convex polyiamond, each subslice is a slice. We have the following result.

Theorem 10 (Polyiamond slices theorem) *The perimeter of a slice-convex polyiamond is the number of slices.*

In considering max polyiamonds, we first consider polyiamonds with given HAD dimensions and max area. Using the Polyiamond HAD max area algorithm (Theorem 11) and the Polyiamond HAD max area formula (Theorem 12), we show that area is maximized for a given perimeter $p = h + a + d$ iff the dimensions are *balanced* (as nearly equal as possible). We consider polyiamonds that are slice-convex, balanced, and complete (filled) in each dimension.

Note that we can rotate and reflect a polyiamond so that $h \leq a \leq d$. For example, the polyiamond of Figure 6 satisfies these inequalities because its HAD dimensions are $(h, a, d) = (6, 7, 8)$. We assume $h \leq a \leq d$ for the rest of the paper. We also assume that $p \neq 1$ or 2, because there are no polyiamonds with these perimeters.

Theorem 11 (Polyiamond HAD max area algorithm) *To construct a polyiamond with given HAD dimensions (h, a, d) and max area, do the following:*

- *Construct a parallelogram with a antidiagonal slices and d diagonal slices.*
- *Pick the h horizontal slices with max area.*

Such a polyiamond is unique, ignoring rotation and reflection.

PROOF. See Figure 7. Note that a polyiamond with a antidiagonal slices and d diagonal slices fits inside a unique parallelogram with a antidiagonal slices and d diagonal slices (this parallelogram is the "AD parallelogram hull", analogous to the convex hull). The parallelogram has height $a + d$, so we must have $h \leq a + d$. For max area, the polyiamond must have no gaps. Constructing the polyiamond as described ensures no gaps, max area, and uniqueness.

Theorem 12 (Polyiamond HAD max area formula) *Let $h \leq a \leq d$. Let*

$$A = ad + ah + dh - \frac{1}{2}(a^2 + d^2 + h^2).$$

The max area of a polyiamond with HAD dimensions (h, a, d) is

$$\operatorname{maxArea}(h,a,d) = \begin{cases} 2ah & h-(d-a)<0 \\ A & h-(d-a)\geq 0 \textit{ and is even} \\ A-1/2 & h-(d-a)\geq 0 \textit{ and is odd} \end{cases}$$

PROOF. There are 3 cases.

- Case: $h-(d-a)<0$. So $h<d-a$. See Figure 7. Each of the $d-a$ horizontal slices in the middle of the parallelogram has max area $2a$. Pick h of these slices to construct a polyiamond with area $2ah$.

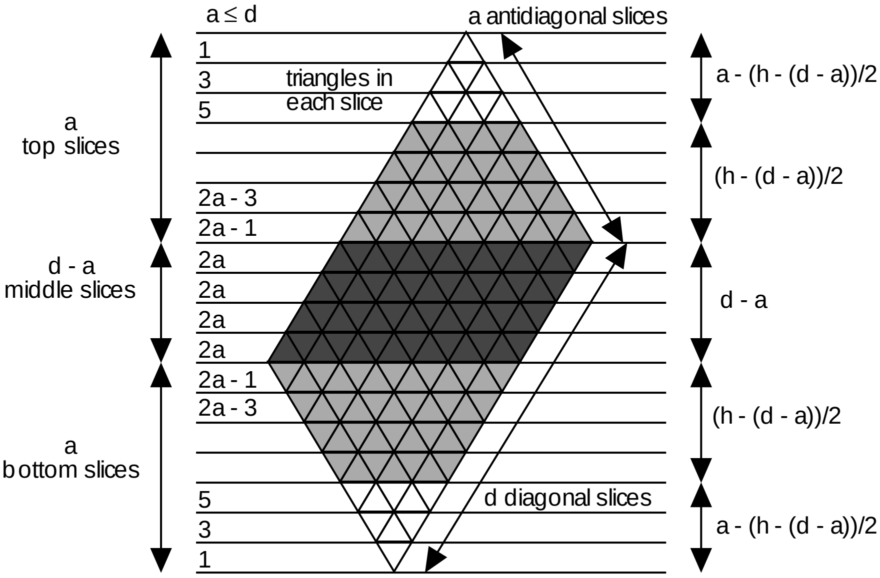

Fig. 7. Calculating the max area of a polyiamond with HAD dimensions (h,a,d), where $h-(d-a)\geq 0$ and is even.

- Case: $h-(d-a)\geq 0$ and is even. Note $h\geq d-a$. See Figure 7, in which the 3 central, shaded sets of horizontal slices have total height h. Consider the top a rows, made up of an unshaded set of a_1 rows (not part of the polyiamond), and a shaded set of $a_2=(h-(d-a))/2$ rows (part of the polyiamond). The area of the top part of the polyiamond is $a(a+1)/2-a_1(a_1+1)/2$. It is easy to verify Table 2.
- Case: $h-(d-a)\geq 0$ and is odd. Note $h\geq d-a$. See Figure 8. It is easy to verify Table 3.

polyiamond part	area
top	$a^2 - \left(a - \frac{h-(d-a)}{2}\right)^2$
middle	$2a(d-a)$
bottom	$a^2 - \left(a - \frac{h-(d-a)}{2}\right)^2$
total	$ad + ah + dh - \frac{1}{2}(a^2 + d^2 + h^2)$

Table 2
Calculating max area of a polyiamond with HAD dimensions (h, a, d), where $h - (d-a) \geq 0$ and is even.

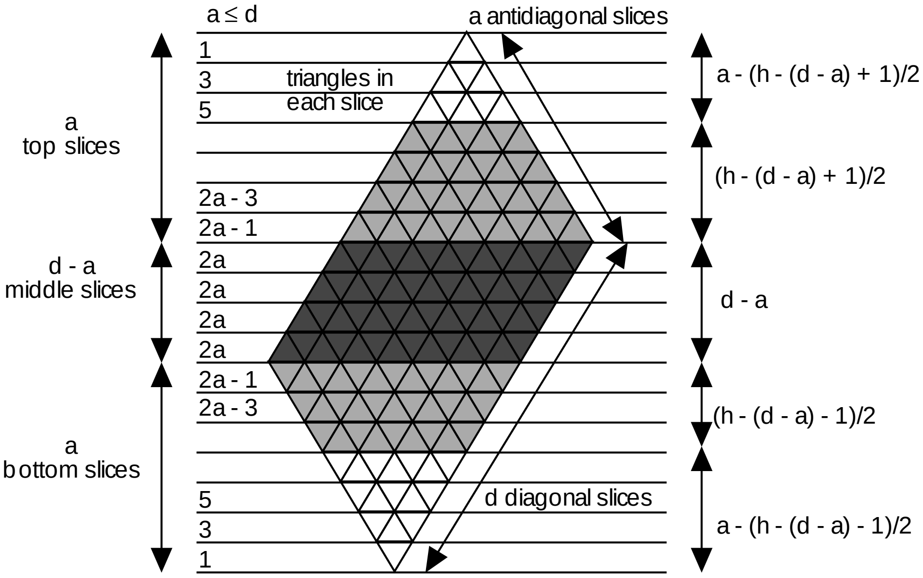

Fig. 8. Calculating the max area of a polyiamond with HAD dimensions (h, a, d), where $h - (d-a) \geq 0$ and is odd.

polyiamond part	area
top	$a^2 - \left(a - \frac{h-(d-a)+1}{2}\right)^2$
middle	$2a(d-a)$
bottom	$a^2 - \left(a - \frac{h-(d-a)-1}{2}\right)^2$
total	$ad + ah + dh - \frac{1}{2}(a^2 + d^2 + h^2) - \frac{1}{2}$

Table 3
Calculating max area of a polyiamond with HAD dimensions (h, a, d), where $h - (d-a-1) \geq 0$ and is odd.

7 Max polyiamond formula

Recall that there is no polyiamond with perimeter 1 or 2.

Theorem 13 (Max polyiamond formula) *Let $p \geq 3$. The max area of a polyiamond with perimeter p is*

$$\mathrm{maxArea}\,(p) = \mathrm{round}\left(\frac{p^2}{6}\right) - \begin{cases} 0 & p \equiv 0 \pmod 6 \\ 1 & \textit{else} \end{cases}$$

PROOF. We will give an integer programming problem that maximizes the area of a polyiamond with perimeter p. We note the following:

- The polyiamond must have no gaps. By the Polyiamond slices theorem (Theorem 10), $h + a + d = p$.
- We can rotate and reflect the polyiamond so that $h \leq a \leq d$.
- We must have $h \geq d - a$, by the following reasoning. If $h < d - a$, then there is some diagonal slice that does not intersect the h horizontal slices of the polyiamond (see Figure 7). Replace this diagonal slice by a horizontal slice. The polyiamond now has HAD dimensions $(h + 1, a, d - 1)$, has more area, and has the same perimeter, contradicting maximality.
- Because $h \geq d-a$, by the Polyiamond HAD max area formula (Theorem 12), the area of the polyiamond is as follows, where $c(h, a, d)$ is a correction term.

$$\text{area} = ad + ah + hd - \frac{1}{2}(a^2 + d^2 + h^2) + c(h, a, d)$$

$$c(h, a, d) = \begin{cases} 0 & h - (d - a) \text{ even} \\ -1/2 & \text{else} \end{cases}$$

The following integer program IP maximizes the area of a polyiamond with perimeter p. Note that p is constant and (h, a, d) is the vector of variables.

$$\mathrm{IP} = \begin{bmatrix} \max\ ad + ah + hd - \frac{1}{2}(a^2 + d^2 + h^2) + c(h, a, d) & & \\ \text{s.t.} \qquad h + a + d = p & & \\ h \leq a & & \leq d \\ h, a, d \in \mathbf{Z} & & \end{bmatrix}$$

Use the constraint $h + a + d = p$ to rewrite the IP objective function as

$$\frac{1}{2}p^2 - (a^2 + d^2 + h^2) + c(h, a, d).$$

In IP, drop the constraint $h \le a \le d$ and the integer constraints. Express the resulting relaxed problem RP in terms of $x^T = (h, a, d)$ and $e^T = (1, 1, 1)$.

$$\text{RP} = \begin{bmatrix} \max \tfrac{1}{2}p^2 - x^T x + c(x) \\ \text{s.t.} \qquad e^T x = p \end{bmatrix} = \frac{p^2}{2} - \begin{bmatrix} \min x^T x - c(x) \\ \text{s.t.} \quad e^T x = p \end{bmatrix}$$

We will derive an alternative expression of the correction term $c(x) = c(h, a, d)$. At the beginning of this proof, we derived $h \ge d - a$. So $h - (d - a) \ge 0$. Note $h - (d - a)$ is even iff $h + a + d = e^T x = p$ is even. We can express the correction term $c(h, a, d)$ as follows:

$$c(h, a, d) = \begin{cases} 0 & h - (d - a) \text{ even} \\ -1/2 & \text{else} \end{cases} = \begin{cases} 0 & p \text{ even} \\ -1/2 & \text{else} \end{cases}$$

The relaxed problem RP branches into 2 relaxed problems, one for the case p odd and one for the case p even.

- Case: p is even. RP has the following form:

$$\text{RP} = \frac{p^2}{2} - \begin{bmatrix} \min x^T x \\ \text{s.t.} \quad e^T x = p \end{bmatrix}$$

  The objective function $x^T x$ is strictly convex, so any solution is unique. To solve RP, use Lagrange multipliers; set the gradient of the objective function equal to the gradient of the constraint.

$$\nabla(x^T x) = \lambda \nabla(e^T x) \implies 2x^T = \lambda e^T \implies x^T = \frac{\lambda}{2} e^T. \implies x = \frac{\lambda}{2} e.$$

  To satisfy the constraint $e^T x = p$, let $\lambda = 2p/3$. So $x_{\text{RP}} = (p/3)e = (p/3, p/3, p/3)$ solves RP. Let $\text{RP}(x_{\text{RP}})$ be the objective value. There are 3 cases: p has the form $6q$, $6q + 2$, or $6q + 4$. x_{RP} is not always integer, and so it does not always solve IP. It turns out that by appropriately rounding the components of x_{RP}, we can get a point $x_{\text{IP}} = (x_{\text{IP}}(h), x_{\text{IP}}(a), x_{\text{IP}}(d))$ that solves IP. (Note that for arbitrary integer programs, component rounding does not guarantee optimality, or even feasibility.) x_{IP} is optimal because $\text{IP}(x_{\text{IP}}) = \lfloor \text{RP}(x_{\text{RP}}) \rfloor$; we omit the straightforward calculations. Table 4 summarizes the results.

- Case: p is odd. RP has the following form:

$$\mathrm{RP} = \frac{p^2}{2} - \begin{bmatrix} \min \frac{1}{2}x^Tx - (-\frac{1}{2}) \\ \text{s.t.} \quad e^Tx = p \end{bmatrix} = \frac{p^2}{2} - \begin{bmatrix} \min \frac{1}{2}x^Tx \\ \text{s.t.} \quad e^Tx = p \end{bmatrix} - \frac{1}{2}$$

There are 3 cases: p has the form $6q+1$, $6q+3$, or $6q+5$. This case is similar to the case in which p is even. Table 4 summarizes the results.

p	$x_{\mathrm{IP}}(h)$	$x_{\mathrm{IP}}(a)$	$x_{\mathrm{IP}}(d)$	$\mathrm{IP}(x_{\mathrm{IP}})$
$6q$	$2q$	$2q$	$2q$	$6q^2$
$6q+1$	$2q$	$2q$	$2q+1$	$6q^2+2q-1$
$6q+2$	$2q$	$2q+1$	$2q+1$	$6q^2+4q$
$6q+3$	$2q+1$	$2q+1$	$2q+1$	$6q^2+6q+1$
$6q+4$	$2q+1$	$2q+1$	$2q+2$	$6q^2+8q+2$
$6q+5$	$2q+1$	$2q+2$	$2q+2$	$6q^2+10q+3$

Table 4
Solutions and optimums of integer problem for maximizing area of a polyiamond with perimeter p, where $p \neq 1$ or 2.

Table 4 and straightforward calculations verify that the max area of a polyiamond with perimeter $p \geq 3$ is

$$\mathrm{maxArea}\,(p) = \mathrm{round}\left(\frac{p^2}{6}\right) - \begin{cases} 0 & p \equiv 0 \pmod 6 \\ 1 & \text{else} \end{cases}$$

Theorem 14 (Max polyiamond balance theorem) *A polyiamond is max iff it has no gaps and is balanced (its dimensions are as nearly equal as possible).*

PROOF. Use Table 4 and the proof of the the Max polyiamond formula (Theorem 13).

We can interpret the max polyiamond formula as approximating a regular hexagon, by the following reasoning. Note $p^2/6$ is integer iff p is a multiple of 6, say $p = 6k$. A regular hexagon of side $p = 6k$ has side k and has $6k^2 = p^2/6$ triangles. If p is not a multiple of 6, then we cannot construct a regular hexagon and we have to subtract 1 triangle from the polyiamond.

8 Max/min polyiamond spiral algorithm

In the previous section, in the Max polyiamond balance theorem (Theorem 14), we gave a slice-based construction that proved that a polyiamond is max iff it has no gaps and is balanced (its dimensions are as nearly equal as possible). In this section, we consider a spiral algorithm for constructing both max and min polyiamonds.

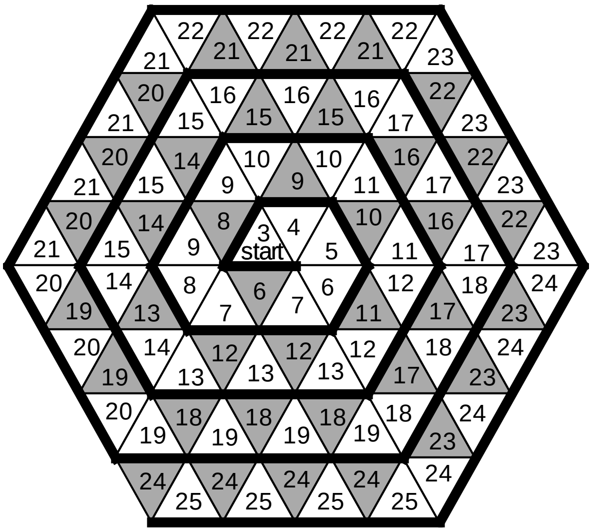

Fig. 9. Max/min polyiamond spiral algorithm. The number in a triangle is the perimeter of the polyiamond constructed. White triangles (except for the first) indicate an increase of 1 in perimeter, and black triangles indicate a decrease of 1 in perimeter.

Theorem 15 (Max/min polyiamond spiral algorithm) *See Figure 9.*

- **Max polyiamond**: *Let* $p \neq 1$ *or* 2. *To construct a max polyiamond with perimeter* p*, follow the spiral until the last appearance of perimeter* p.
- **Min polyiamond**: *To construct a min* n*-polyiamond, follow the spiral for* n *triangles.*

PROOF. [Proof of max polyiamond part]

In Table 4, note that the dimensions (h, a, d) of a max polyiamond with perimeter p increase as follows: d increases first, then a, then h, ... We can interpret this increasing of dimensions (and the corresponding increasing of area) as starting with a polyiamond with HAD dimensions $(h, a, d) = (2q, 2q, 2q)$ and then adding triangles in a clockwise spiral; we complete 1 revolution of the spiral by successively filling new slices in the following order: d, a, h, d, a, h; it is as if we are traveling around a hexagonal clock.

It is easy to verify that we use all of the "space" in the current dimension, thereby achieving the "capacity" shown by the value of $\mathrm{IP}(x_{\mathrm{IP}})$ in Table 4. The appropriate dimension (h, a, or d) is increased and filled by the next set of added triangles. See Figure 9.

PROOF. [Proof of min polyiamond part]

This follows from the following claim.

> Claim: Let minPerimeter(n) be the min perimeter of an n-polyiamond. Let r be the min perimeter of a polyiamond with area at least n. (Think of r as the "perimeter required for area at least n".) Then minPerimeter(n) is whichever of r or $r+1$ has the same parity as n.

Proof of claim: Note minPerimeter(n) $\geq r$. We improve this lower bound for minPerimeter(n) as follows. By the Polyiamond perimeter parity (Theorem 5), minPerimeter(n) has the same parity as n. There are 2 cases. If r has the same parity as n, then minPerimeter(n) $\geq r$. Else, $r+1$ has the same parity as n, and minPerimeter(n) $\geq r+1$. So an improved lower bound for minPerimeter(n) is whichever of r or $r+1$ has the same parity as n.

We prove that this improved lower bound is attainable by constructing an n-polyiamond whose perimeter is this improved lower bound. See Figure 10. Follow the spiral for $\mathrm{maxArea}(r-1)$ triangles to construct the unique max polyiamond with perimeter $r-1$. Let $1 \leq \Delta \leq \mathrm{maxArea}(r) - \mathrm{maxArea}(r-1)$. Follow the spiral for Δ more triangles to construct a polyiamond with area $\mathrm{maxArea}(r-1) + \Delta$. Note $n = \mathrm{maxArea}(r-1) + \Delta$ for some Δ.

In Figure 10, we can see that for $\Delta = 1$, the perimeter of the polyiamond constructed is r, and for $\Delta = 2$, the perimeter is $r+1$. In general, the perimeters alternate between r and $r+1$. So the perimeter of the n-polyiamond constructed is r or $r+1$. By the discussion at the beginning of this proof, the perimeter is actually minPerimeter(n), and is whichever of r or $r+1$ has the same parity as n.

9 Min polyiamond bounds

When we proved the Min polyiamond formula (Theorem 7), we assumed the min polyiamond bounds, which we prove in this section.

Theorem 16 (Min polyiamond intermediate perimeter theorem) *Let $k \geq 1$. Table 5 gives the perimeters of the min polyiamonds constructed by*

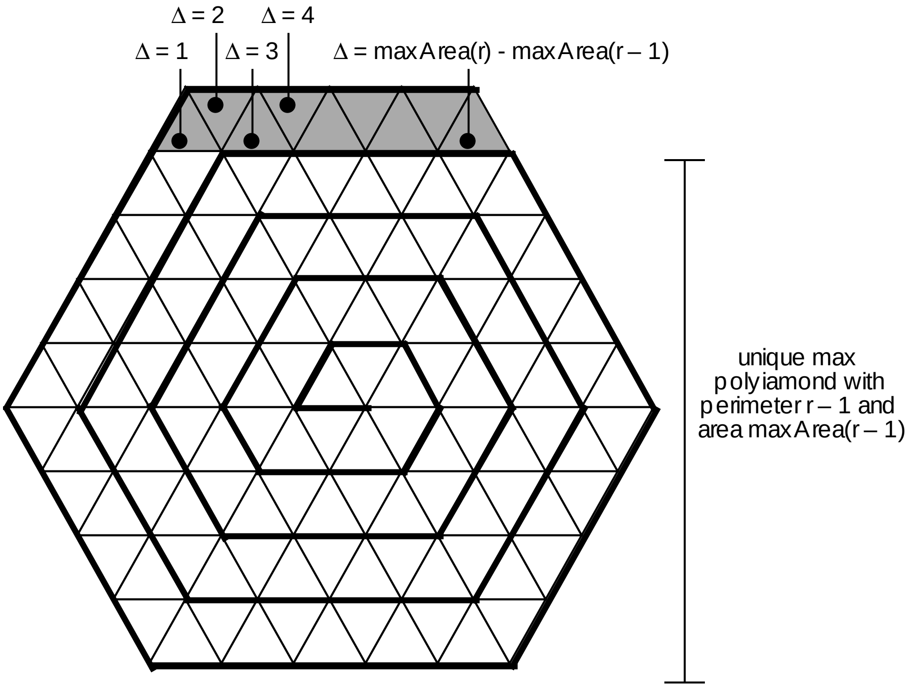

Fig. 10. Perimeter of polyiamond constructed by adding Δ triangles to the unique max polyiamond with perimeter $r-1$ and area maxArea $(r-1)$.

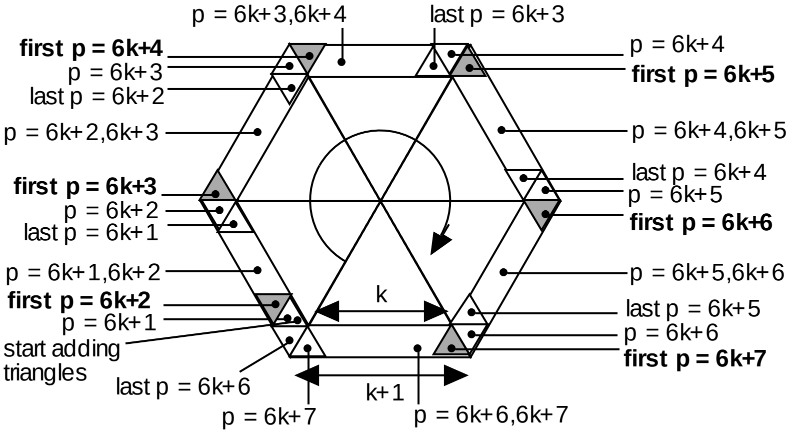

Fig. 11. Appearances of perimeters in the Max/min polyiamond spiral algorithm (Theorem 15).

adding Δ triangles to a regular hexagon of side k in the Max/min polyiamond spiral algorithm (Theorem 15).

PROOF. See Figure 11, which just keeps track of which triangles increase or decrease the perimeter in the Max/min polyiamond spiral algorithm (Theorem 15).

Theorem 17 (Min polyiamond ceiling theorem) *Let $k \geq 1$. Let $n =$*

Δ	perimeter
$0 \le \Delta \le 0$	$6k$
$1 \le \Delta \le 2k$	$6k + \{1, 2\}$
$2k+1 \le \Delta \le 4k+1$	$6k + \{2, 3\}$
$4k+2 \le \Delta \le 6k+2$	$6k + \{3, 4\}$
$6k+3 \le \Delta \le 8k+3$	$6k + \{4, 5\}$
$8k+4 \le \Delta \le 10k+4$	$6k + \{5, 6\}$
$10k+5 \le \Delta \le 12k+6$	$6k + \{6, 7\}$

Table 5
Perimeters of min polyiamonds constructed by adding Δ triangles to a regular hexagon of side k in the Max/min polyiamond spiral algorithm (Theorem 15).

$6k^2 + \Delta$. *Table 6 gives the values of* $\left\lceil \sqrt{6n} \right\rceil$.

Δ	$\lceil \sqrt{6n} \rceil$
$0 \le \Delta \le 0$	$6k$
$1 \le \Delta \le 2k$	$6k + 1$
$2k+1 \le \Delta \le 4k$	$6k + 2$
$4k+1 \le \Delta \le 6k+1$	$6k + 3$
$6k+2 \le \Delta \le 8k+2$	$6k + 4$
$8k+3 \le \Delta \le 10k+4$	$6k + 5$
$10k+5 \le \Delta \le 12k+6$	$6k + 6$

Table 6
Values of $\lceil \sqrt{6n} \rceil$, where $n = 6k^2 + \Delta$.

PROOF. The case $\Delta = 0$ is trivial. For the other cases, use the following abbreviated calculations with $i = 1, \ldots, 5$.

$$\begin{array}{rrcl} & & \left\lceil \sqrt{6n} \right\rceil & = 6k + i \\ \Longleftrightarrow & 6k + (i-1) < & \sqrt{6n} & \le 6k + i \\ \Longleftrightarrow & 2k(i-1) + \dfrac{(i-1)^2}{6} < & \Delta & \le 2ki + \dfrac{i^2}{6}. \end{array}$$

Theorem 18 (Min polyiamond bounds) *The perimeter of an min n-polyiamond satisfies the following bounds:*

$$\left\lceil \sqrt{6n} \right\rceil \leq \mathrm{minPerimeter}(n) \leq \left\lceil \sqrt{6n} \right\rceil + 1.$$

PROOF. Use the Max/min polyiamond spiral algorithm (Theorem 15). There are 2 cases.

- Case: $n \leq 6$. It is easy to verify the bounds for these values of n.
- Case: $n > 6$. Compare Table 5 and Table 6. Δ is the number of triangles added to a regular hexagon of side k, and $n = 6k^2 + \Delta$. The intervals of Δ in the tables are very similar; the endpoints differ by at most 1. It is easy to check that the bounds hold for all values of Δ.

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