# Celebrating Fifty Years of David M. Young's Successive Overrelaxation Method

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## Abstract

It has been over fifty years since David M. Young's original work on the successive overrelaxation (SOR) methods. This fundamental method now appears in all textbooks containing an introductory discussion of iterative solution methods. (Most often the SOR method appears after a presentation of Jacobi iteration and Gauss-Seidel iteration and before the conjugate gradient iterative method.) We present a brief survey of some of the research of Professor David M. Young, together with his students and collaborators, on iterative methods for solving large sparse linear algebraic equations. This is not a complete survey but just a sampling of various papers with a focus on some of these publications.

Dr. David M. Young's doctoral thesis *Iterative methods for solving partial difference equations of elliptic type* was accepted in 1950 by his supervising professor Garrett Birkhoff, Harvard University, and his landmark paper by the same name appeared in *Transactions of the American Mathematics Society*, Volume 76, pp. 92–111, 1954. He will celebrate his 80th birthday on October 20, 2003.

*Key words:* successive overrelaxation (SOR) method, Property A, Chebyshev acceleration, conjugate gradient acceleration, nonsymmetric systems, software for iterative methods, alternating-type iterative methods,

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## 1 Introduction

We present a brief survey of some of the work of Professor David M. Young, together with his students and collaborators, on iterative methods. Dr. David M. Young has been involved in research on iterative methods for solving large sparse linear algebraic equations for over forty years until his recent retirement. This is not a complete survey but just a sampling of various projects with a focus on some of his publications.

*Preprint submitted to Elsevier Science* *14 February 2003*

## 2 Successive Overrelaxation

From research first done at Harvard University, Young presented in his Ph.D. thesis (Young, 1950), and in a subsequent paper (Young, 1954), an analysis of the successive overrelaxation (SOR) method for the case where the coefficient matrix of the linear algebraic system $Au = b$ is consistently ordered (Young, 1971, 2003). An elliptic partial differential equations over a region with grid points numbered in the *natural ordering* (left-to-right and up) and using the standard five-point discretization stencil results in such a matrix system. In fact, any matrix system derived in this way has Young's *Property A* (Young, 1971, 2003). Moreover, a consistently ordered system can be obtained from one with Property A after a suitable permutation.

For a matrix with Property A, one can permute the rows and corresponding columns to obtain a *red-black system.* The *red-black ordering* corresponds to a red and black checkerboard ordering of the grid points. When $A$ is a red-black matrix, it is consistently ordered and Young's equation (Young, 1950) $(\lambda+\omega-1)^2 = \omega^2\mu^2\lambda$ gives a relation between the eigenvalues $\lambda$ of the iteration matrix $\mathcal{L}_\omega$ for the SOR method and the eigenvalues $\mu$ of the iteration matrix $B$ for the Jacobi method. If $A$ is symmetric positive definite, then the eigenvalues of $B$ are real and less than 1 in absolute value and the optimum or *best* value of the acceleration factor $\omega$ is given by $\omega_b = 2/\left(1 + \sqrt{1 - S(B)^2}\right)$. Here $S(B)$ is the *spectral radius* of the Jacobi matrix $B$, which is the magnitude of the eigenvalue of largest absolute value of the matrix $B$. Moreover, the spectral radius of the SOR matrix with the optimum relaxation parameter $\omega = \omega_b$ is given by $S(\mathcal{L}_{\omega_b}) = \omega_b - 1 = r$.

For model problems involving the Poisson equation over a region with mesh points of grid size $h$, it can be shown that the number of iterations required for convergence of the SOR method is $n = \mathcal{O}(h^{-1})$ whereas the number of iterations is $n = \mathcal{O}(h^{-2})$ using either the Jacobi or Gauss-Seidel methods. In this situation, the SOR method is faster by an order of magnitude.

Work has been done on the choice of the optimum $\omega$ for the case where $A$ is consistently ordered but not symmetric positive definite and where some of the eigenvalues of the Jacobi iteration matrix $B$ are *complex eigenvalues.* Several programs are available for choosing the optimum $\omega$ if all of the eigenvalues of $B$ are known or if one knows a convex region containing them. See Young and Eidson (1970) and Young and Huang (1983).

An extension of the SOR method is the *modified SOR (MSOR) method* for a linear system with a red-black coefficient matrix. The MSOR method involves the use of relaxation factors $\omega_1, \omega_1'$, $\omega_2, \omega_2'$, …, where $\omega_i$ is used for the *red* components, and $\omega_i'$ is used for the *black* points, for each $i$. In Young, Downing, and Wheeler (1965), it is shown that there are suitable values of $\omega_i$ and $\omega_i'$ that are as good, though not better than, the choice $\omega_i = \omega_i' = \omega_b$ for all $i$. On the other hand, other choices are more effective if one measures the effectiveness in terms of certain norms as shown in Young and Kincaid (1969). Chapters 8 and 10 of Young (1971, 2003) cover the modified SOR method with fixed and variable parameters, respectively.

A number of other modifications and extensions have been made to the SOR theory. For instance in *group* or *block methods*, the unknowns are grouped into blocks and all values within a block are updated simultaneously. Usually, each inner iteration of a block method is done by a direct method since the matrices for the blocks are assumed to be easily solvable. For example, these matrices are tridiagonal in the case of the line SOR method when the five-point finite difference stencil is used. Also, faster convergence is obtained for line SOR methods than for point SOR methods, in general. Moreover, the general SOR theory has been applied to group iterative methods in Chapter 14 of Young (1971, 2003).

Research on *norms* associated with the SOR method for the red-black system has resulted in new formulas. It has been shown that graph of the $D^{\frac{1}{2}}$-norm function for the SOR matrix $\mathcal{L}_{\omega_b}^m$ is not monotonically decreasing (it increases and then decreases), but the $A^{\frac{1}{2}}$-norm is indeed a monotonically decreasing function of $m$; however, it is still considerably larger than the spectral radius function $S(\mathcal{L}_{\omega_b}^m) = r^m$. See Young and Kincaid (1969) and Chapter 7 in Young (1971, 2003).

Corresponding to the SOR method is the *Symmetric Successive Overrelaxation (SSOR) method* in which an iteration consists of one iteration of the (forward) SOR method followed by one iteration of the (backward) SOR method. In the *Unsymmetric Successive Overrelaxation (USSOR) method*, different parameters may be used in the red and black equations, respectively. Young (1970) presents convergence properties of the symmetric and unsymmetric successive overrelaxation methods and related methods.

## 3 Chebyshev Acceleration

The SOR method can be regarded as a way to accelerate the convergence of the Jacobi method in a certain sense. Another way of speeding up the convergence of the Jacobi method is to use an extrapolation method or a Chebyshev acceleration method, which is based on Chebyshev polynomials. These are general procedures and they can be applied to methods other than just the Jacobi method as shown in Hageman and Young (1981, 2003).

Suppose the basic iterative method to be used in the acceleration procedure has the form $u^{(n+1)} = Gu^{(n)} + k$ where the eigenvalues $\mu$ of $G$ are bounded such that $m(G) \leq \mu \leq M(G)$. Here $m(G)$ and $M(G)$ are the smallest and largest eigenvalues of $G$, respectively. Using the three-term relation for Chebyshev polynomials, the optimal *Chebyshev acceleration method* can be written as $u^{(n+1)} = \rho_{n+1}\{\gamma(Gu^{(n)} + k) + (1 - \gamma)u^{(n)}\} + (1 - \rho_{n+1})u^{(n-1)}$ where $\rho_{n+1} = (1 - (\sigma/2)^2\rho_n)$, (with $\rho_1 = 1$ and $\rho_2 = (1 - \sigma^2/2)$), $\sigma = [M(G) - m(G)]/[2 - M(G) - m(G)]$, and $\gamma = 2/[2 - M(G) - m(G)]$.

Varga (1962, 2000) refers to this procedure as the *Chebyshev semi-iterative method*. One needs to choose estimates for $M(G)$ and $m(G)$, which may cause difficulties in some cases. In fact, the behavior of the acceleration procedure is often sensitive to these estimates and especially the one for $M(G)$. It can be shown that the optimum Chebyshev acceleration procedure is an order of magnitude faster than the optimum extrapolated procedure for $\sigma$ close to one (Hageman and Young, 1981, 2003).

If one applies the Chebyshev acceleration procedure to the Jacobi method as the basic method for solving a linear system with a red-black coefficient matrix, then the computation can be simplified. This is done by rewriting the procedure in terms of only the red points or only the black points for the Jacobi method. Golub and Varga (1961) refer to this as the *cyclic Chebyshev semi-iterative method*. The cyclic acceleration method is the original Chebyshev acceleration method with half of the calculations bypassed. The resulting method is equivalent to a special case of the modified SOR method with $\omega_i = \omega'_i = \rho_{n+1}$.

With an *adaptive Chebyshev acceleration procedure*, one continuously revises the iteration parameters as the iterative method proceeds. The algorithm fixes the smallest eigenvalue estimate $m_E \leq m(G)$ and adaptively modifies the largest eigenvalue estimate $M_E$ but keep $M_E \leq M(G)$. The iterative procedure continues using these values of $m_E$ and $M_E$ until the observed convergence is much slower than expected in a certain sense. By solving a *Chebyshev equation*, the algorithm increases $M_E$ but keeps $M_E \leq M(G)$. This adaptive Chebyshev acceleration procedure is repeated until convergence is achieved according to the stopping test being utilized. Chebyshev polynomials are used in the algorithm for choosing these maximum and minimum eigenvalue estimates. Such a procedure was developed by Hageman and Young (1981, 2003) and it was incorporated into the algorithms used in the ITPACK software packages (Kincaid, Respess, Young, and Grimes, 1982).

It has been shown that, in some cases, one can obtain almost as good a convergence as with Chebyshev acceleration by the use of a *stationary second degree method* given by $u^{(n+1)} = \rho\{\gamma(Gu^{(n)} + k) + (1 - \gamma)u^{(n)}\} + (1 - \rho)u^{(n-1)}$. Here let $\rho = 1$ when $n = 0$. See some of the papers on second-degree methods are Kincaid (1974, 1994); Kincaid and Young (1993); Young (1972).

## 4 Conjugate Gradient Acceleration

*Conjugate gradient acceleration* $u^{(n+1)} = \rho_{n+1}\{\gamma_{n+1}(r^{(n)}+u^{(n)}\}+(1-\rho_{n+1})u^{(n-1)}$ is similar to Chebyshev acceleration except that the parameters used $\rho_{n+1} = [1-(\gamma_{n+1}/\gamma_n\rho_n)\langle r^{(n)}, r^{(n)}\rangle/\langle r^{(n-1)}, r^{(n-1)}\rangle]^{-1}$, (with $\rho_1 = 1$) and $\gamma_{n+1} = \langle r^{(n)}, r^{(n)}\rangle/\langle r^{(n)}, Ar^{(n)}\rangle$ involve inner products. Here $r^{(n)} = b - Au^{(n)}$ is the residual vector. As with Chebyshev acceleration method, conjugate gradient acceleration method can speed-up the Jacobi method and other methods. Conjugate gradient acceleration has some advantages over Chebyshev acceleration (Hageman and Young, 1981, 2003). It can be shown that the convergence of conjugate gradient acceleration, measured in a certain norm, is at least as fast as that of Chebyshev acceleration. With conjugate gradient acceleration there are no parameter estimates; however, the basic iterative method may involve a parameter as in the case when SSOR is used as the basic method. Since the conjugate gradient acceleration requires the computation of inner products for each iteration, the work required per iteration may be somewhat greater than for Chebyshev acceleration. For basic methods that are not symmetrizable, the generalized conjugate gradient methods can be used to accelerate their convergence (Hageman and Young, 1981, 2003).

We can introduce a nonsingular matrix $W$ as follows $[W(I-G)W^{-1}][Wu] = Wk$ which in terms of the original linear system is $[WQ^{-1}AW^{-1}][Wu] = WQ^{-1}b$. The *generalized conjugate gradient acceleration method* (Hageman and Young, 1981, 2003) corresponding to the basic iterative method is given by $u^{(n+1)} = \rho_{n+1}\{\gamma_{n+1}\delta^{(n)} + u^{(n)}\} + (1-\rho_{n+1})u^{(n-1)}$ where $\rho_{n+1} = [1 - (\gamma_{n+1}/\gamma_n\rho_n)\langle W\delta^{(n)}, W\delta^{(n)}\rangle/\langle W\delta^{(n-1)}, W\delta^{(n-1)}\rangle]^{-1}$, (with $\rho_1 = 1$), $\gamma_{n+1} = [1 - \langle W\delta^{(n)}, WG\delta^{(n)}\rangle/\langle W\delta^{(n)}, W\delta^{(n)}\rangle]^{-1}$, and the *pseudo-residual vector* is $\delta^{(n)} = Gu^{(n)} + k - u^{(n)}$. The conjugate gradient acceleration method minimizes the $[W^TW(I-G)]^{\frac{1}{2}}$-matrix-norm of the error as compared with any polynomial acceleration procedure based. If $A$ and $Q$ are symmetric positive definite matrices and if $W^TW = Q$, then we minimize the $A^{\frac{1}{2}}$-matrix-norm of the error as in the conjugate gradient method. It can be shown that the average rate of convergence for the conjugate gradient method, when measured in the $[W^TW(I-G)]^{\frac{1}{2}}$-matrix-norm, is at least as large as that for the corresponding Chebyshev acceleration procedure. (See Hageman and Young (1981, 2003).)

## 5 Nonsymmetric Systems

A difficult problem is solving the linear system when the coefficient matrix $A$ is not necessarily symmetric positive definite or even symmetric. Three *generalized conjugate gradient acceleration methods* called ORTHODIR, ORTHOMIN, and ORTHORES were considered by Young and Jea (1980). It was shown that under fairly general conditions these methods converge, in exact arithmetic, in at most $N$ iterations, where $N$ is the order of the matrix. Also in Jea and Young (1983), the *biconjugate gradient (BCG) method* as well as other forms of *Lanczos methods* were considered as generalized conjugate gradient acceleration methods corresponding to certain double linear systems involving $A$ and $A^T$.

The *generalized minimum residual (GMRES) method* (Saad and Schultz, 1986) is a widely used method for solving nonsymmetric linear systems. The method is generally very reliable although stagnation may occur in some cases. Moreover, for nonsymmetric systems, the amount of work required per iteration usually increases as the number of iterations increases. ‘Recently, Young working with Chen and Kincaid developed various generalizations of the GMRES method and combined them with the Lanczos procedure. New iterative methods called GGMRES, MGMRES, and LAN/MGMRES have been established. (See Chen, Kincaid, and Young (1998a), Chen (1997); Chen, Kincaid, and Young (1998b), and Chen, Kincaid, and Young (1999).) The GGMRES method is a slight generalization of the GMRES method. The GMRES method minimizes a norm of the residual and GGMRES minimizes a more general norm—both involve a minimization condition. Alternatively, the MGMRES method is a modification of the GMRES method applied to a symmetric indefinite linear system using a Galerkin condition. This latter method is related to the BCG method and to other variants of the Lanczos method. The LAN/GMRES method aims at combining the reliability of the GMRES method with the reduced work of a Lanczos-type method. When conducting initial numerical experiments on nonsymmetric linear systems arising from convection-diffusion problems, it was found that LAN/MGMRES was comparable with a number of other methods that are extensively used. Further tests over a wider class of problems are planned to obtain a better understanding of these methods with the objective of improving them.

## 6 Software for Iterative Methods

Under the direction of Kincaid and Young, several research-oriented software packages were written as part of the ITPACK Project at the Center for Numerical Analysis. Beginning in the mid-1970s, there was an increased effort to develop iterative algorithms and portable public domain software. Software packages, such as the ITPACK 2C package, were developed, which included automatic procedures for handling choices that were causing difficulties for users of iterative methods. Automatic procedures included were developed for determining all necessary iteration parameters and for accurate and realistic stopping tests for iterative algorithms. Algorithms based on these procedures were described in the book by Hageman and Young (1981, 2003). Also, software from the ITPACK 2C package was modified and incorporated into the ELLPACK package at Purdue University for solving elliptic partial differential equations. (See Rice and Boisvert (1985).)

While the ITPACK 2C package was intended primarily for linear systems where the coefficient matrix is symmetric positive definite or nearly so, other packages such as NSPCG (Oppe, Joubert, and Kincaid, 1988) and PCG (Joubert, 1996) were developed with the capability of handling nonsymmetric systems. Other ITPACK Project software include ITPACKV 2D (Kincaid, Oppe, and Young, 1986), ITPACK 3A (Young and Mai, 1984) and ITAPCK 3B (Mai and Young, 1986), for example. See Kincaid and Young (1988) for a review of the ITPACK software packages.

## 7 Alternating-Type Iterative Methods

To construct an alternating-type method for solving $Au = b$, we choose matrices $H, V$, and $\Sigma$ such that $A = H + V + \Sigma$, where $\Sigma$ is a diagonal matrix with positive diagonal elements. For any linear system of the form $(H + \rho\Sigma)v = w$ or $(V + \rho\Sigma)v = w$, we assume that $H + \rho\Sigma$ and $V + \rho\Sigma$ are nonsingular matrices for any positive real number $\rho$ and so that for any vector $w$ one can easily solve for $v$. To define an *alternating-type iterative method*, we choose positive numbers $\rho$ and $\rho'$ and, for a given $u^{(n)}$, we determine $u^{(n+\frac{1}{2})}$ and $u^{(n+1)}$ by $(H+\rho\Sigma)u^{(n+\frac{1}{2})} = b-(V-\rho\Sigma)u^{(n)}$ and $(V+\rho'\Sigma)u^{(n+1)} = b-(H-\rho'\Sigma)u^{(n+\frac{1}{2})}$. Thus, we have $u^{(n+1)} = T_{\rho,\rho'}u^{(n)} + k_{\rho,\rho'}$ where $T_{\rho,\rho'} = (V + \rho'\Sigma)^{-1}(H - \rho'\Sigma)(H + \rho\Sigma)^{-1}(V - \rho\Sigma) = I - (\rho + \rho')(V + \rho'\Sigma)^{-1}\Sigma(H + \rho\Sigma)^{-1}(H + V)$ and $k_{\rho,\rho'} = (\rho + \rho')(V + \rho'\Sigma)^{-1}\Sigma(H + \rho\Sigma)^{-1}b = (I - T_{\rho,\rho'})A^{-1}b$. Examples of alternating-type methods are the *alternating-direction implicit (ADI) method*, the *symmetric successive overrelaxation (SSOR) method*, and the *unsymmetric successive overrelaxation (USSOR) method*. With the ADI method, $H$ and $V$ are either tridiagonal or are permutationally similar to tridiagonal matrices and $\Sigma = I$. With the SSOR and USSOR methods, $H$ and $V$ are lower triangular and upper triangular matrices, respectively, and $\Sigma$ is a diagonal matrix with positive diagonal elements.

In certain cases, the ADI method converges rapidly. For example, with problems involving Poisson's equation in the rectangle with a grid of mesh size

$h$, the ADI method convergences in $n = \mathcal{O}(\log h^{-1})$ iterations using the optimum number of parameters and in $n = \mathcal{O}(h^{-1/m})$ iterations using the best $m$ parameters. Recall that $n = \mathcal{O}(h^{-1})$ for the SOR method. The *commutative case* is when $HV = VH$, $H\Sigma = \Sigma H$, $V\Sigma = \Sigma V$. It holds for certain separable self-adjoint elliptic partial differential equations defined over rectangles. Given the commutativity condition and also bounds on the eigenvalues of $H$ and $V$, necessary and sufficient convergence conditions related to choosing ADI parameters can be found in Birkhoff, Varga, and Young (1962) and in Chapter 17 of Young (1971, 2003). Also see Young and Wheeler (1964).

With a *nonstationary alternating-type iterative method*, the parameters $\rho$ and $\rho'$ may vary from iteration to iteration. We seek to determine the parameters $\{\rho_i\}$ and $\{\rho'_i\}$ so that $u^{(m)}$ is as close to the true solution $\overline{u} = A^{-1}b$ as possible. In practice, we seek to make the spectral radius $S\big(\prod_{i=1}^{m} T_{\rho_i,\rho'_i}\big)$ as small as possible. As an alternative to the (sequential) non-stationary method, we consider the *parallel alternating-type iterative method*. See papers by Young and Kincaid (1995, 1996).

## 8 Books

Professor Young wrote the classical research monograph *Iterative Solution of Large Linear Systems* (Young, 1971, 2003) and then the book *Applied Iterative Methods* (Hageman and Young, 1981, 2003) on iterative algorithms was written in collaboration with Louis A. Hageman. Also, we should mention the two volume textbook *A Survey of Numerical Mathematics* co-authored with Robert T. Gregory (Young and Gregory, 1972, 1988, 1973, 1988).

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