On Convergence Rate of the Augmented Lagrangian Algorithm for Nonsymmetric Saddle Point Problems

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We are interested in solving the system

$$\begin{pmatrix} A & L^T \\ L & 0 \end{pmatrix} \begin{pmatrix} c \\ \lambda \end{pmatrix} = \begin{bmatrix} F \\ G \end{bmatrix} \tag{1}$$

by a variant of the augmented lagrangian algorithm. This type of problem typically arises in certain discretizations of the Navier-Stokes equations. Here A is a (n, n) matrix, c, $F \in \mathbb{R}^n$, L is a (m, n) matrix, and $\lambda, G \in \mathbb{R}^m$. We assume that A is invertible on the kernel of L. Convergence rates of the augmented lagrangian algorithm are known in the symmetric case but the proofs in [Glowinski and LeTallec'89] used spectral arguments and cannot be extended to the nonsymmetric case. The purpose of this paper is to give a rate of convergence of a variant of the algorithm in the nonsymmetric case.

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1 Introduction

We will use the same notations $(.,.)$ and $\|\cdot\|$ for the inner products and norms in $\mathbb{R}^n$ and $\mathbb{R}^m$. The particular inner product will be identified by the types of matrices appearing. The augmented lagrangian algorithm for symmetric problems can be derived by minimization arguments. We refer to [Glowinski and LeTallec'89] for details. It is described as follows: $r > 0$ and $\rho_l > 0$ for all l are parameters and M is a preconditionner. Given $\lambda^{(0)} \in \mathbb{R}^m$ specified arbitrarily, with $\lambda^{(l)}$ known, compute $c^{(l)}$ then $\lambda^{(l+1)}$ by

$$\begin{cases} (A + rL^T L)c^{(l)} + L^T\lambda^{(l)} = F + rL^T G \\ \lambda^{(l+1)} = \lambda^{(l)} + \rho_l M^{-1}(Lc^{(l)} - G). \end{cases} \tag{2}$$

In [Awanou and Lai'02], we were interested in a variant of this algorithm for $\rho_l = \rho = \frac{1}{\epsilon}$ for all l, $r = \frac{1}{\epsilon}$ where $\epsilon > 0$ is fixed and M taken to be the identity matrix. More precisely, for this choice of the parameters, the algorithm reads

$$\begin{cases} (A + \frac{1}{\epsilon}L^T L)c^{(l)} + L^T\lambda^{(l)} = F + \frac{1}{\epsilon}L^T G \\ \lambda^{(l+1)} = \lambda^{(l)} + \frac{1}{\epsilon}(Lc^{(l)} - G). \end{cases} \tag{3}$$

The variant we considered is the following algorithm

$$\begin{cases} (A + \frac{1}{\epsilon}L^T L)c^{(l+1)} + L^T\lambda^{(l)} = F + \frac{1}{\epsilon}L^T G \\ \lambda^{(l+1)} = \lambda^{(l)} + \frac{1}{\epsilon}(Lc^{(l+1)} - G), \end{cases} \tag{4}$$

which can be easily shown to be equivalent to the following sequence of problems

$$\begin{pmatrix} A & L^T \\ L & -\epsilon I \end{pmatrix} \begin{pmatrix} c^{(l+1)} \\ \lambda^{(l+1)} \end{pmatrix} = \begin{bmatrix} F \\ G - \epsilon\lambda^{(l)}, \end{bmatrix} \tag{5}$$

where I is the identity matrix of size $m \times m$. In [Gunzburger'89], it was claimed that the later algorithm converges to the solution c of (1) and

$$\|c - c^{(l+1)}\| \leq C\epsilon\|c - c^{(l)}\|,$$

for a constant $C > 0$. We have not however been able to find a proof of this result in the literature. The main objective of this article is to prove the convergence of the following algorithm

$$\begin{cases} (A + rL^T L)c^{(l+1)} + L^T\lambda^{(l)} = F + rL^T G \\ \lambda^{(l+1)} = \lambda^{(l)} + \rho M^{-1}(Lc^{(l+1)} - G), \ \rho > 0, \end{cases} \tag{6}$$

which generalizes (4) and give a convergence rate similar to the one above. A fine study of the convergence rate still appears to be difficult (cf. remark 2.12 p. 64 [Glowinski and LeTallec'89]).

The paper is organized as follows. We first give a sufficient condition for solvability of (1) which leads to a Ladyzhenskaya–Babuška–Brezzi (LBB) type condition. We then prove the convergence rate.

2 Solvability

In this section, we derive a sufficient condition for the solvability of (1). Let Ker(X) and Im(X) denote the kernel and range of the operator X. We first give a few lemmas:

Lemma 1 $\mathbb{R}^n = \mathrm{Ker(L)} \oplus \mathrm{Im(L^T)}$ *and* $\mathbb{R}^m = \mathrm{Ker(L^T)} \oplus \mathrm{Im(L)}$.

Proof: It is enough to prove only one of the decompositions. Since $\mathrm{Im(L)} \subseteq \mathbb{R}^{\mathrm{m}}$, we have $\mathbb{R}^m = \mathrm{Im(L)} \oplus \mathrm{Im(L)}^\perp$, where $\mathrm{Im(L)}^\perp$ denotes the orthogonal of Im(L). We need to show that

$$\mathrm{Im(L)}^\perp = \mathrm{Ker(L^T)}.$$

Let $q \in \mathrm{Im(L)}^\perp$. For $w \in \mathbb{R}^n$, $Lw \in \mathrm{Im(L)}$, so $q^T(Lw) = 0$. Therefore $w^T L^T q = 0$ so that $L^T q$ is orthogonal to $\mathbb{R}^n$, that is $L^T q = 0$, i.e. $q \in \mathrm{Ker(L^T)}$. This argument also shows that $\mathrm{Ker(L^T)} \subset \mathrm{Im(L)}^\perp$ and the result follows. The

following result can be found in [Bramble, Pasciak and Vassilev'99], we give here a detailed proof for convenience.

Lemma 2 *Suppose A is an invertible linear operator with positive definite symmetric part A_s that satisfy*

$$(Ax, y) \leq \alpha (A_s x, x)^{\frac{1}{2}} (A_s y, y)^{\frac{1}{2}}, \quad \textit{for all } x, y \in \mathbb{R}^n \textit{ and } \alpha \geq 1,$$

then $(A^{-1})_s$ is positive definite and satisfy

$$((A^{-1})_s w, w) \leq ((A_s)^{-1} w, w) \leq \alpha^2 (A^{-1})_s w, w) \quad \textit{for all } w \in \mathbb{R}^n.$$

Moreover

$$(A^{-1}x, y) \leq \Big((A_s)^{-1}x, x\Big)^{\frac{1}{2}} \Big((A_s)^{-1}y, y\Big)^{\frac{1}{2}}.$$

Proof:

$$\Big((A_s)^{-1}w, w\Big) = \sup_{y \in \mathbf{R}^n} \frac{(w, y)^2}{(A_s y, y)},$$

and

$$(w, y)^2 = (w, A^{-1}Ay)^2 = ((A^{-1})^T w, Ay)^2 \leq \alpha^2 (A_s y, y)(A_s (A^{-1})^T w, (A^{-1})^T w)$$

by (2). So

$$\begin{aligned}(w, y)^2 \leq \alpha^2 ||y||^2_{A_s} (A(A^{-1})^T w, (A^{-1})^T w) &= ||y||^2_{A_s} ((A^{-1})^T w, A^T (A^{-1})^T w) \\ &= ||y||^2_{A_s} ((A^{-1})^T w, w),\end{aligned}$$

where $||.||^2_{A_s} = (A_s.,.)$ and we used $(Aw, w) = (A_s w, w)$ for all $w \in \mathbb{R}^n$. It follows that

$$\Big((A_s)^{-1}w, w\Big) \leq \alpha^2 (A^{-1})_s w, w).$$

On the other hand (using fractional power of symmetric positive definite matrices which can be defined via singular values decomposition)

$$\begin{aligned}((A^{-1})_s w, w) &= (A^{-1}w, w) = (A_s^{\frac{1}{2}} A^{-1} w, (A_s)^{-\frac{1}{2}} w) \leq ||A_s^{\frac{1}{2}} A^{-1} w|| \;\; ||(A_s)^{-\frac{1}{2}} w|| \\ &= ||A^{-1}w||_{A_s} ||w||_{(A_s)^{-1}} = (AA^{-1}w, A^{-1}w)^{\frac{1}{2}} ||w||_{(A_s)^{-1}} \\ &= (A^{-1}w, w)^{\frac{1}{2}} ||w||_{(A_s)^{-1}}.\end{aligned}$$

It follows that

$$((A^{-1})_s w, w) \leq \Big((A_s)^{-1} w, w\Big).$$

In addition

$$\begin{aligned}(A^{-1}x, y) &= (A_s^{\frac{1}{2}} A^{-1} x, (A_s)^{-\frac{1}{2}} y) \\ &\leq (A^{-1}x, x)^{\frac{1}{2}} \Big((A_s)^{-1} y, y\Big)^{\frac{1}{2}} \quad \text{using the same arguments as above} \\ &\leq \Big((A_s)^{-1} x, x\Big)^{\frac{1}{2}} \Big((A_s)^{-1} y, y\Big)^{\frac{1}{2}} \quad \text{using (2)}\ .\end{aligned}$$

We can now prove the following theorem

Theorem 3 *Let A be a matrix which satisfies the condition (2) and has a symmetric part A_s positive definite with respect to L in the sense that $x^T A_s x \geq 0$ and $x^T A_s x = 0$ with $Lx = 0$ implies $x = 0$, then (1) is solvable and moreover*

$$sup_{u \in \mathbb{R}^n} \frac{(y, Lu)^2}{(A_r v, v)} \geq c_1 ||y||^2 \quad \textit{for all } y \in Im(L), \tag{7}$$

where $c_1 > 0$ is a constant which depends on r, $A_r = A + rL^T L$ and $Im(L)$ denotes the range of L.

Proof: We have

$$\begin{cases} Ac + L^T \lambda \ = F \\ \qquad\quad Lc = G, \end{cases} \tag{8}$$

so $rL^T Lc = rL^T G$ which gives

$$(A + rL^T L)c + L^T \lambda = F + rL^T G.$$

We first show that $A + rL^T L$ is invertible. Since A is a square matrix, it is enough to show that

$$(A + rL^T L)x = 0 \Rightarrow x = 0.$$

We have

$$x^T(A+rL^TL)x = x^T(A_s+rL^TL)x = x^TA_sx + r(Lx)^T(Lx),$$

so by the assumptions on A,

$$x^T(A+rL^TL)x = 0 \Rightarrow x^TA_sx = 0 \text{ and } (Lx)^T(Lx) = 0.$$

It follows that $x^TA_sx = 0$ and $Lx = 0$. Since A_s is assumed to be symmetric positive definite with respect to L, we get $x = 0$. We can therefore write

$$c = (A+rL^TL)^{-1}(F+rL^TG) - (A+rL^TL)^{-1}L^T\lambda.$$

Since $Lc = G$, we see that the solvability of (1) is equivalent to solving

$$L(A+rL^TL)^{-1}L^T\lambda = L(A+rL^TL)^{-1}(F+rL^TG) - G$$

for λ. By Lemma (1), we have $\lambda = \lambda_0 + \overline{\lambda}$ with $\lambda_0 \in \mathrm{Ker}(L^T)$ and $\overline{\lambda} \in \mathrm{Im}(L)$. Clearly it is enough to find $\overline{\lambda}$. We show that there is $c_1 > 0$ such that

$$y^TL(A+rL^TL)^{-1}L^Ty \geq c_1||y||^2 \quad \text{for all } y \in \mathrm{Im}(L).$$

This will imply that $B(A+rL^TL)^{-1}L^T$ is invertible on $\mathrm{Im}(L)$ and show that (1) is solvable.

Since A satisfies (2) and because

$$(A_sx, x) \leq ((A_s+rL^TL)x, x) \quad \text{and} \quad (rL^TLx, x) \leq ((A_s+rL^TL)x, x),$$

we have

$$((A+rL^TL)x, y) \leq \alpha((A_s+rL^TL)x, x)^{\frac{1}{2}}(A_s+rL^TL)y, y)^{\frac{1}{2}}, \quad \text{for all } x, y \in \mathbb{R}^n.$$

It follows from Lemma (2) that

$$((A+rL^TL)_s^{-1}w, w) \geq \frac{1}{\alpha^2}((A_s+rL^TL)^{-1}w, w) \quad \text{for all } w \in \mathbb{R}^n.$$

Because A_s is positive definite with respect to L, $A_s + rL^TL$ is symmetric positive definite and so $L(A_s+rL^TL)^{-1}L^T$ is symmetric positive definite on $\mathrm{Im}(L)$ ($L^Tz = 0$ and $z \in \mathrm{Im}(L)$ implies $z = 0$) so that we have

$$z^TL(A_s+rL^TL)^{-1}L^Tz \geq c_0||z||^2, \quad \text{for all } z \in \mathrm{Im}(L),$$

with $c_0 > 0$ depending on r. This gives

$$\begin{aligned}
y^T L(A + rL^T L)^{-1} L^T y &= (L^T y)^T (A + rL^T L)^{-1} (L^T y) \\
&= (L^T y)^T (A + rL^T L)_s^{-1} (L^T y) \\
&\geq \frac{1}{\alpha^2} (L^T y)^T (A_s + rL^T L)^{-1} (L^T y) \\
&= \frac{1}{\alpha^2} y^T L(A_s + rL^T L)^{-1} L^T y \\
&\geq \frac{c_0}{\alpha^2} ||y||^2, \quad \text{for all } y \in \text{Im}(L).
\end{aligned}$$

We have therefore proved (2) with $c_1 = \frac{c_0}{\alpha^2}$ and have completed the proof that (1) is solvable.

Next, we show that (2) is equivalent to (7) which is a LBB type condition. It is enough to notice that

$$\begin{aligned}
y^T L A_r^{-1} L^T y &= (A_r^{-1} L^T y, L^T y) \\
&= \sup_{u \in \mathbf{R}^n} \frac{(L^T y, u)^2}{(A_r u, u)} \\
&= \sup_{u \in \mathbf{R}^n} \frac{(y, Lu)^2}{(A_r u, u)}.
\end{aligned}$$

3 Convergence

In this section, we prove the convergence of the iterative algorithm (6). In the next section we give the convergence rate.

Theorem 4 *Suppose that the linear system (1) has a unique solution c and that A_s the symmetric part of A is positive definite with respect to L. Moreover, assume that M is symmetric positive definite. Then, the sequence $(c^{(l)})$ defined in (6) converges to the solution c of (1) for $r \geq \frac{\rho ||M||}{2}$.*

Proof: Clearly (6) is solvable since A_r is invertible. With $\lambda^{(l)}$ given one computes successively $c^{(l+1)}$ and $\lambda^{(l+1)}$.

The original problem (1),

$$\begin{cases} Ac + L^T \lambda = F \\ \qquad\quad Lc = G, \end{cases}$$

can be rewritten as

$$\begin{cases} (A + rL^T L)c + L^T \lambda = F + rL^T G \\ \qquad\qquad\qquad\qquad \lambda = \lambda + \rho M^{-1}(Lc - G). \end{cases} \tag{9}$$

Let $u^{(l+1)} = c^{(l+1)} - c$ and $p^{(l+1)} = \lambda^{(l+1)} - \lambda$. We have, using (9) and (2),

$$(A + rL^T L)u^{(l+1)} + L^T p^{(l)} = 0 \tag{10}$$

and

$$p^{(l+1)} = p^{(l)} + \rho M^{-1} L u^{(l+1)}. \tag{11}$$

We deduce from (11) that

$$\|p^{(l+1)}\|_M^2 = (Mp^{(l+1)}, p^{(l+1)}) = (Mp^{(l)} + \rho L u^{(l+1)}, p^{(l)} + \rho M^{-1} L u^{(l+1)})$$

which gives

$$\begin{aligned} \|p^{(l+1)}\|_M^2 &= \|p^{(l)}\|_M^2 + (Mp^{(l)}, \rho M^{-1} L u^{(l+1)}) \\ &\quad + (\rho L u^{(l+1)}, p^{(l)}) + \rho^2 (L u^{(l+1)}, M^{-1} L u^{(l+1)}) \\ &= \|p^{(l)}\|_M^2 + 2\rho(p^{(l)}, L u^{(l+1)}) + \rho^2 (L u^{(l+1)}, M^{-1} L u^{(l+1)}), \end{aligned}$$

since M^{-1} is symmetric and hence

$$\|p^{(l)}\|_M^2 - \|p^{(l+1)}\|_M^2 = -2\rho(p^{(l)}, L u^{(l+1)}) - \rho^2 (L u^{(l+1)}, M^{-1} L u^{(l+1)}). \tag{12}$$

It follows from (10) that

$$(Au^{(l+1)}, u^{(l+1)}) + r\|L u^{(l+1)}\|^2 = -(p^{(l)}, L u^{(l+1)}),$$

and hence by substituing this into (12), we get

$$\begin{aligned} \|p^{(l)}\|_M^2 - \|p^{(l+1)}\|_M^2 &= 2\rho(Au^{(l+1)}, u^{(l+1)}) + 2r\rho\|L u^{(l+1)}\|^2 \\ &\quad - \rho^2 (L u^{(l+1)}, M^{-1} L u^{(l+1)}). \end{aligned}$$

Next, we notice that

$$\begin{aligned} (L u^{(l+1)}, M^{-1} L u^{(l+1)}) &\leq ||L u^{(l+1)}||_M \; ||M^{-1} L u^{(l+1)}||_M \\ &= ||L u^{(l+1)}||_M \; (L u^{(l+1)}, M^{-1} L u^{(l+1)})^{\frac{1}{2}}, \end{aligned}$$

so

$$\begin{aligned} (L u^{(l+1)}, M^{-1} L u^{(l+1)}) &\leq ||L u^{(l+1)}||_M^2 \\ &= (M L u^{(l+1)}, L u^{(l+1)}) \leq ||M L u^{(l+1)}|| \; ||L u^{(l+1)}|| \\ &\leq ||M|| \; ||L u^{(l+1)}||^2. \end{aligned}$$

Therefore

$$\|p^{(l)}\|_M^2 - \|p^{(l+1)}\|_M^2 \geq 2\rho(Au^{(l+1)}, u^{(l+1)}) + \Big(2r\rho - ||M||\ \rho^2\Big)||Lu^{(l+1)}||^2.$$

So, provided $r \geq \frac{\rho\|M\|}{2}$ since A_s is nonnegative,

$$\|p^{(l)}\|_M^2 - \|p^{(l+1)}\|_M^2 \geq 0,$$

i.e.

$$\|p^{(l+1)}\|_M \leq \|p^{(l)}\|_M, \quad \text{for all } l$$

and the sequence $\{\|p^{(l)}\|_M\}$ is seen to be decreasing. Being bounded below by 0, it converges; hence $\|p^{(l)}\|^2 - \|p^{(l+1)}\|^2$ converges to 0 which implies that $(A_s u^{(l)}, u^{(l)})$ and $\|Lu^{(l)}\|^2$ converge to 0. Since $A_s + \frac{1}{\epsilon}L^T L$ is positive definite, it follows that $u^{(l)}$ converges to 0 and finally $c^{(l)}$ converges to c.

4 Convergence Rate

To show the convergence rate, we will need the following lemma.

Lemma 5 . *The mappings* $L : \mathrm{Im(L^T)} \rightarrow \mathrm{Im(L)}$ *and* $L^T : \mathrm{Im(L)} \rightarrow \mathrm{Im(L^T)}$ *are bijections with bounded inverses.*

Proof: We show that L is one-one on $\mathrm{Im(L^T)}$. This is immediate since $Lx = 0$ with $x \in \mathrm{Im(L^T)}$ implies $x \in \mathrm{Im(L^T)} \cap \mathrm{Ker(L)} = \{0\}$ by Lemma (1). As a linear mapping between finite dimensional spaces, L has a bounded inverse on Im(L) and there exists $k_0 > 0$ such that for any $g \in \mathrm{Im(L)}$, there exists $v_g \in \mathrm{Im(L^T)}$ such that

$$\|v_g\| \leq \frac{1}{k_0}\|g\|. \tag{13}$$

A similar proof applies to L^T. This completes the proof.

We would like to elaborate on this last inequality. Let $v \in \mathbb{R}^n$ and $g = Lv \in \mathrm{Im(L)}$ so there is $v_g \in \mathrm{Im(L^T)}$ for which $g = Lv_g$. That is, $Lv = Lv_g$. It follows that $v_g = v + v_0$ for some $v_0 \in \mathrm{Ker(L)}$. We therefore have

$$\|v + v_0\| \leq \frac{1}{k_0}\|Lv\| = \frac{1}{k_0}\|L(v + v_0)\|$$

for any $v + v_0 \in \mathrm{Im(L^T)}$. It follows that

$$\|v\| \leq \frac{1}{k_0}\|Lv\| = \frac{1}{k_0}\sup_{q\in\mathbf{R}^n}\ \frac{(q, Lv)}{\|q\|}, \quad \text{for all } v \in \mathrm{Im(L^T)}.$$

The same arguments applied to L^T show that there is $k_1 > 0$ such that

$$\|q\| \leq \frac{1}{k_1}\|L^T q\| = \frac{1}{k_1}\sup_{v\in\mathbf{R}^n}\frac{(v, L^T q)}{\|v\|} \quad \text{for all } q \in \mathrm{Im(L)}.$$

We have the following theorem

Theorem 6 *Suppose that the linear system (1 has a unique solution c and that A_s the symmetric part of A is positive definite with respect to L. Moreover, assume that M is symmetric positive definite. Then,*

$$\|c - c^{(l+1)}\| \leq C\epsilon\|c - c^{(l)}\|,$$

for a constant C which depends on r and ρ. Moreover for $r = \rho = \frac{1}{\epsilon}$ and $M = I$,

$$\|c - c^{(l+1)}\| \leq C\epsilon\|c - c^{(l)}\|,$$

for a positive constant C independent of l and ϵ.

Proof: Solving for $u^{(l+1)}$ in (10) and substituting in (11), we get

$$p^{(l+1)} = (I - \rho M^{-1}L(A + rL^T L)^{-1}L^T)p^{(l)}.$$

It follows that $p^{(l+1)} - p^{(l)} = \lambda^{(l+1)} - \lambda^{(l)}$ is in the range of L. Since

$$\lambda^{(k+1)} - \lambda = \sum_{j=1}^{k+1}(\lambda^{(j)} - \lambda^{(j-1)}) + (\lambda^{(0)} - \lambda) = \sum_{j=1}^{k+1} p^{(j)} - p^{(j-1)} + p^{(0)},$$

we have $\lambda^k - \lambda \in \mathrm{Im(L)}$ for each k since we may assume $p^{(0)} \in \mathrm{Im(L)}$. We write $u^{(l+1)} = \hat{u}^{(l+1)} + \overline{u}^{(l+1)}$ with $\hat{u}^{(l+1)} \in \mathrm{Ker}(L)$ and $\overline{u}^{(l+1)} \in \mathrm{Im}(L^T)$. Using (4), we have

$$k_0\|\overline{u}^{(l+1)}\| \leq \sup_{q\in\mathbf{R}^n}\frac{q^T L u^{(l+1)}}{\|q\|},$$

where we used the inner product $(M.,.)$ instead of the canonical one. Using (11), we have

$$q^T L u^{(l+1)} = \frac{1}{\rho}q^T M p^{(l+1)} - \frac{1}{\rho}q^T M p^{(l)},$$

so

$$k_0\|\overline{u}^{(l+1)}\| \leq \frac{||M||}{\rho}\left(\|p^{(l+1)}\|_M + \|p^{(l)}\|_M\right).$$

By (3) we have $\|p^{(l+1)}\|_M \leq \|p^{(l)}\|_M$ which gives

$$\|\overline{u}^{(l+1)}\| \leq \frac{2||M||}{\rho k_0}\|p^{(l)}\|_M.$$

We next give a bound on $\hat{u}^{(l+1)}$. Since $A + rL^T L$ is invertible, A is invertible on $\mathrm{Ker}(L)$. Indeed if $Ax = 0$ and $Lx = 0$, then $(A + rL^T L)x = 0$ which implies that $x = 0$. Therefore there is $\alpha_0 > 0$ such that

$$\alpha_0 \|\hat{u}^{(l+1)}\| \leq \|A\hat{u}^{(l+1)}\| = \sup_{v_0 \in \mathrm{Ker}(L)} \frac{(v_0, A\hat{u}^{(l+1)})}{\|v_0\|}$$
$$= \sup_{v_0 \in \mathrm{Ker}(L)} \frac{v_0^T A u^{(l+1)} - v_0^T A \overline{u}^{(l+1)}}{\|v_0\|}$$

However, from (10), we have $Au^{(l+1)} = -L^T p^{(l)} - rL^T L u^{(l+1)}$ which implies

$$v_0^T A u^{(l+1)} = -v_0^T L^T p^{(l)} - r v_0^T L^T L u^{(l+1)}$$
$$= -(Lv_0)^T p^{(l)} - r(Lv_0)^T L u^{(l+1)} = 0$$

for $v_0 \in \mathrm{Ker}(L)$. Thus,

$$\alpha_0 \|\hat{u}^{(l+1)}\| \leq \sup_{v_0 \in \mathrm{Ker}(L)} \frac{-v_0^T A \overline{u}^{(l+1)}}{\|v_0\|}$$
$$\leq \|A\| \, \|\overline{u}^{(l+1)}\|$$
$$\leq \frac{2\|A\|}{\rho k_0} \|p^{(l)}\|_M.$$

We therefore have

$$\|u^{(l+1)}\| \leq \|\overline{u}^{(l+1)}\| + \|\hat{u}^{(l+1)}\|$$
$$\leq \frac{2}{\rho k_0} \Big(\frac{||A||}{\alpha_0} + ||M|| \Big) \|p^{(l)}\|_M.$$

We now give a bound on $\|p^{(l)}\|_M$ in terms of $\|u^{(l)}\|$. Since $p^{(l)} \in \mathrm{Im}(L)$, (4) gives

$$k_1 \|p^{(l)}\|_M \leq \sup_{v \in \mathrm{R}^n} \frac{v^T L^T p^{(l)}}{\|v\|}.$$

Combining the equations in (6), we get

$$(A + rL^T L - \rho L^T M^{-1} L) u^{(l)} + L^T p^{(l)} = 0,$$

thus $v^T L^T p^{(l)} = -v^T (A + rL^T L - \rho L^T M^{-1} L) u^{(l)}$. Therefore,

$$\|p^{(l)}\|_M \leq \frac{||A_{r,\rho}||}{k_1} \|u^{(l)}\|,$$

where $A_{r,\rho} = A + rL^T L - \rho L^T M^{-1}$. It follows that

$$\|u^{(l+1)}\| \leq \frac{2}{\rho k_0} \Big(\frac{||A||}{\alpha_0} + ||M|| \Big) \frac{||A_{r,\rho}||}{k_1} \|u^{(l)}\|.$$

For $r = \rho = \frac{1}{\epsilon}$ and $M = I$, $A_{r,\rho} = A$ that is,

$$\|c - c^{(l+1)}\| \leq C\epsilon \|c - c^{(l)}\|,$$

for a positive constant C independent of l and ϵ.

5 Conclusion

In this paper, we have given a convergence rate of a variant of the augmented lagrangian algorithm. We intend to undertake a numerical study of the optimal choice of the parameters r and ρ when this algorithm is applied to the incompressible Navier-Stokes equations filling a gap left by earlier researchers.

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