Review of Differentiation and Integration for MATH 3200
CU-Denver Department of Mathematical Sciences

This is a packet of prerequisite material necessary for understanding material covered in ordinary differential equations. Many students take this course after having taken their previous course many years ago, at another institution where certain topics may have been omitted, or have not seen this material in a long time or just feel uncomfortable with it. Because understanding this material is so important to being successful in this course, we have put together this review packet.

In this packet you will find sample questions and a brief discussion of each topic. If you find the material in this pamphlet is not sufficient for you, it may be necessary for you to look in the appropriate sections of a calculus textbook and understand the material on your own. Because this is considered prerequisite material, it is ultimately your responsibility to learn it. The topics to be covered include Differentiation and Integration.

1 Differentiation

In this course you will be expected to be able to differentiate and integrate quickly and accurately. What follows is a very brief *review*. If you find this is not enough of a review, please refer to your calculus textbook.

Exercises:

1. Find the derivative of $y = x^3 \sin(x)$.
2. Find the derivative of $y = \frac{\ln(x)}{\cos(x)}$.
3. Find the derivative of $y = \ln(\sin(e^{2x}))$.

Discussion:
It is expected that you know, without looking at a table, the following differentiation rules:

$$\frac{d}{dx}\left[(kx)^n\right] = kn(kx)^{n-1} \tag{1}$$

$$\frac{d}{dx}\left[e^{kx}\right] = ke^{kx} \tag{2}$$

$$\frac{d}{dx}\left[\ln(kx)\right] = \frac{1}{x} \tag{3}$$

$$\frac{d}{dx}\left[\sin(kx)\right] = k\cos kx \tag{4}$$

$$\frac{d}{dx}\left[\cos(kx)\right] = -k\sin x \tag{5}$$

$$\frac{d}{dx}[uv] = u'v + uv' \tag{6}$$

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{u'v - uv'}{v^2} \tag{7}$$

$$\frac{d}{dx}[u(v(x))] = u'(v)v'(x). \tag{8}$$

We put in the constant k into (1) - (5) because a very common mistake to make is something like: $\frac{d}{dx}e^{2x} = \frac{e^{2x}}{2}$ (when the correct answer is $2e^{2x}$). Equation (6) is known as the product rule, Equation (7) is known as the quotient rule, and Equation (8) is known as the chain rule. From these, you can derive the derivative of many other functions, such as the tangent:

$$\begin{aligned}
\tan(x) &= \frac{\sin(x)}{\cos(x)} \\
\frac{d}{dx}[\tan(x)] &= \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{(\cos(x))^2} \\
&= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\
&= \frac{1}{\cos^2(x)} \\
&= \sec^2(x)
\end{aligned}$$

where we have used the quotient rule and simplified.

The chain rule is applied when there is a *function of a function*, i.e. $f(g(x))$. The idea is to take the derivative of the outside function first, leaving its argument alone. Then multiply that by the derivative of the next outermost function, leaving it's argument alone. The process is repeated until there is nothing left of which to take the derivative. So for example, to take the derivative of $\sin^2(5x)$, we need to first determine the outside function. If we re-write it as $(\sin(5x))^2$ it is quickly determined that the outside function is "something squared", where "something" in this case is $\sin(5x)$. The derivative of "something squared" is "2 times that something times the derivative of that something". Thus we have

$$\begin{aligned}
\tfrac{d}{dx}\left[\sin^2(5x)\right] &= \frac{d}{dx}\left[(\sin(5x))^2\right] \\
&= 2\sin(5x)\frac{d}{dx}[\sin(5x)] \\
&= 2\sin(5x)\cos(5x)\frac{d}{dx}[5x] \\
&= 2\sin(5x)\cos(5x)5 \\
&= 10\sin(5x)\cos(5x)
\end{aligned}$$

Solution to Exercises:

1. For this problem, we need the product rule, (6), since two functions are being multiplied. In this case, $u(x) = x^3$ and $v(x) = \sin(x)$. Thus,

$$\begin{aligned}\frac{d}{dx}\left[x^3 \sin(x)\right] &= 3x^2 \sin(x) + x^3[\cos(x)] \\ &= 3x^2 \sin(x) + x^3 \cos(x).\end{aligned}$$

2. This is clearly a quotient of functions, so that the quotient rule applies, (7). We have $u(x) = \ln(x)$ and $v(x) = \cos(x)$, which implies:

$$\begin{aligned}\frac{d}{dx}\left[\frac{\ln(x)}{\cos(x)}\right] &= \frac{\frac{1}{x}\cos(x) - \ln(x)[-\sin(x)]}{(\cos(x))^2} \\ &= \frac{\frac{1}{x}\cos(x) + \ln(x)\sin(x)}{\cos^2(x)} \\ &= \frac{\cos(x) + x\ln(x)\sin(x)}{x\cos^2(x)}\end{aligned}$$

3. This is a case of a function of a function of a function of a function, $f(g(h(i(x))))$. We apply the chain rule, always working from the outside function in. In this case the (very) outside function is $f() = \ln$ of "something"; the next most outside function is, $g() = \sin$ of "something", the next most outside function is, $h() = e$ to the "something", and the inside most function is $i(x) = 2x$. Applying the chain rule (8) we have

$$\begin{aligned}\frac{d}{dx}\left[\ln(\sin(e^{2x}))\right] &= \frac{1}{\sin(e^{2x})}\frac{d}{dx}\left[\sin(e^{2x})\right] \\ &= \frac{1}{\sin(e^{2x})}\cos(e^{2x})\frac{d}{dx}\left[e^{2x}\right] \\ &= \frac{1}{\sin(e^{2x})}\cos(e^{2x})e^{2x}\frac{d}{dx}[2x] \\ &= \frac{1}{\sin(e^{2x})}\cos(e^{2x})e^{2x}2 \\ &= \frac{2e^{2x}\cos(e^{2x})}{\sin(e^{2x})} \\ &= 2e^{2x}\cot(e^{2x})\end{aligned}$$

2 Integration

Solving differential equations requires integration - there's just no getting around it. What follows is a brief review. If you need supplemental material, please see your calculus text.

Exercises:

1. Evaluate $\int x\sqrt{x^2+1}\,dx$.

2. Evaluate $\int \frac{\sin(x)}{\cos(x)}\,dx$.

3. Evaluate $\int xe^{3x}\,dx$

4. Given $\int u^n \ln(u)\,du = \frac{u^{n+1}\ln(u)}{n+1} - \frac{u^{n+1}}{(n+1)^2} + C$, evaluate $\int x^2 \ln(2x)\,dx$.

The discussion section is rather long, and the solution to these exercises are given at the end of this section.

Discussion:

2.1 Basic Integration Formulas

Not only is it important to be familiar with various integration techniques, but it is also important that we be quick and efficient when evaluating integrals so that we can concentrate on the concepts as oppose to the mechanics of integration.

Examples

1. We can evaluate the indefinite integral $\int e^{2x}dx$ by doing a *u-substitution*. However, we can become more efficient at evaluating integrals of this type by obtaining a general formula. Let $f(x) = e^{ax}$, where a is equal to a constant. We would like to obtain a general formula for $\int e^{ax}dx$. We can accomplish this by doing a u-substitution. Let $u = ax$, then $du = adx \Rightarrow \frac{du}{a} = dx$. Substitution yields

$$\int e^{ax}dx = \tfrac{1}{a}\int e^{u}du = \tfrac{1}{a}e^{u} + C = \tfrac{1}{a}e^{ax} + C$$

   Now we have a general formula that we can use again and again without going to the trouble of doing the u-substitution each time. For example,

   (a) $y(x) = e^{2x}$

$$\int e^{2x}dx = \frac{1}{2}e^{2x} + C$$

   (b) $y(x) = e^{\pi x}$

$$\int e^{\pi x}dx = \frac{1}{\pi}e^{\pi x} + C$$

2. Let us follow the procedure in *Example 1* to find the general formula for integrals of the form $\int \cos(ax)\ dx$, where a is equal to a constant. Let $u = ax$, then $du = adx \Rightarrow \frac{du}{a} = dx$. Substitution yields

$$\int \cos(ax)\ dx = \tfrac{1}{a} \int \cos(u)\ du = \tfrac{1}{a} \sin(u) + C = \tfrac{1}{a} \sin(ax) + C.$$

Here are some examples of using this general formula.

(a) $y(x) = \cos(4x)$

$$\int \cos(4x)\ dx = \frac{1}{4} \sin(4x) + C$$

(b) $y(x) = \cos \frac{1}{2\pi} x$

$$\int \cos\left(\frac{1}{2\pi} x\right) dx = 2\pi \sin 2\pi x + C$$

Exercises

Follow the examples above to obtain a general formula for the integral given, then use it to evaluate parts (a) and (b). As above, a is equal to a constant.

1. $\int \sin(ax)\ dx$

   (a) $\int \sin(16x)\ dx$

   (b) $\int \sin\left(\frac{1}{2}x\right)\ dx$

2. $\int \ln(ax)\ dx$

   (a) $\int \ln(\pi x)\ dx$

   (b) $\int \ln(\frac{1}{\pi}x)\ dx$

3. $\int \tan(ax)\ dx$

   (a) $\int \tan(3x)\ dx$

   (b) $\int \tan\left(\frac{1}{3}x\right)\ dx$

4. $\int \sec(ax)\ dx$

(a) $\int \sec(2.78x)\ dx$

(b) $\int \sec(1618x)\ dx$

5. $\int \arctan(ax)\ dx$

(a) $\int \arctan(\pi x)\ dx$

(b) $\int \arctan\left(\frac{1}{\pi}x\right)\ dx$

2.2 u-substitution

In general, u-substitutions are not as straight forward as the ones in the previous section. When doing a u-substitution you want to look for the part of the integral whose derivative is elsewhere in the integral (up to a constant). Formally, if we have an integral of the form

$$\int f(g(x))g^{'}(x)dx,$$

we let $u = g(x)$, then $du = g^{'}(x)dx$, substitution yields

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Essentially, we have transformed the space in which we are evaluating the integral. We evaluate the integral in this new space and then substitute u back in to obtain a solution in the original space. Similar techniques are often employed to solve differential equations.

Examples

1. $\int x^5 e^{x^6} dx$
First, let $u = x^6$, then $du = 6x^5 \Rightarrow \frac{du}{6} = x^5$. Substitution yields

$$\begin{aligned} \int x^5 e^{x^6} dx &= \frac{1}{6}\int e^u du \\ &= \frac{1}{6}e^u + C \\ &= \frac{1}{6}e^{x^6} + C. \end{aligned}$$

2. $\int (x^2+1)^2(2x)\ dx$
First, let $u = x^2 + 1$, then $du = 2x\ dx$. Substitution yields

$$\begin{aligned}\int (x^2+1)^2(2x)dx &= \int u^2\,du \\ &= \frac{1}{3}u^3 + C \\ &= \frac{1}{3}(x^2+1)^3 + C.\end{aligned}$$

Exercises

Use u-substitution to evaluate the following indefinite integrals.

1. $\int \sin^2(3x)\cos(3x)\,dx$
2. $\int \frac{1}{\theta^2}\cos\left(\frac{1}{\theta}\right)\,d\theta$
3. $\int \frac{\sin x}{\cos^2 x}\,dx$
4. $\int e^x(e^x+1)^2\,dx$
5. $\int \tan^4 x\sec^2 x\,dx$

2.3 Integration by Parts

Integration by parts is applicable to a plethora of functions which we may need to integrate. Formally, if u and v are functions of x and have continuous derivatives, then

$$\int udv = uv - \int vdu$$

Choosing which part is equal to u may be facilitated by remembering the acronym: LIATE, which stands for: Logarithm, Inverse trig, Algebraic, Trigonometric, Exponential. This means that whichever of these expressions appears first in the acronym, that is the expression you should let u be. So if you want to evaluate $\int(x^2+5x-2)e^{5x}\,dx$, we see we have an algebraic expression, x^2+5x-2, times an exponential function, e^{5x}. By this acronym, since A appears before E, we set $u = x^2+5x-2$.

Examples

1. Evaluate $\int xe^x\,dx$
   First, let $u = x \Rightarrow du = dx$ and let $dv = e^x dx \Rightarrow v = e^x$. Using the integration by parts formula we obtain

$$\begin{aligned}\int xe^x\,dx &= xe^x - \int e^x dx \\ &= xe^x - e^x + C.\end{aligned}$$

2. Evaluate $\int \arcsin x dx$
First, let $u = \arcsin x \Rightarrow du = \frac{1}{\sqrt{1-x^2}} dx$ and let $dv = dx \Rightarrow v = x$.

$$\begin{aligned}
\int \arcsin x dx &= x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx \\
&= x \arcsin x + \frac{1}{2} \int w^{-\frac{1}{2}} dw \\
&= x \arcsin x + w^{\frac{1}{2}} + C \\
&= x \arcsin x + (1-x^2)^{\frac{1}{2}} + C \\
&= x \arcsin x + \sqrt{1-x^2} + C.
\end{aligned}$$

Where the second equality comes from doing a $u-substitution$ (w in this case) where $w = 1 - x^2$

3. Sometimes it is necessary to do integration by parts more than once. For example, $\int x^2 e^x dx$.
First, let $u = x^2 \Rightarrow du = 2xdx$ and let $dv = e^x dx \Rightarrow v = e^x$. Substitution yields

$$\begin{aligned}
\int x^2 e^x dx &= x^2 e^x - 2 \int x e^x dx \\
&= x^2 e^x - 2(xe^x - e^x) + C \\
&= x^2 e^x - 2xe^x - 2e^x + C.
\end{aligned}$$

where the second equality comes from our previous calculation in *Example 1.*

4. Here is another example where integration by parts will be used repeatedly to evaluate an integral. Evaluate $y(x) = e^x \cos x$.
First, let $u = e^x \Rightarrow du = e^x dx$ and let $dv = \cos x \; dx \Rightarrow v = \sin x$. Substitution yields

$$\int e^x \sin x \; dx = e^x \sin x - \int e^x \sin x dx.$$

Using integration by parts again, let $u = e^x \Rightarrow du = e^x dx$ and let $dv = \sin x \; dx \Rightarrow v = -\cos x$. Substitution yields

$$\begin{aligned}
\int e^x \cos x \; dx &= e^x \sin x - \left[-e^x \cos x + \int e^x \cos x \; dx \right] \\
&= e^x \sin x + e^x \cos x - \int e^x \cos x \; dx \\
2 \int e^x \cos x \; dx &= e^x \sin x + e^x \cos x \\
\int e^x \cos x \; dx &= \frac{1}{2}(e^x \sin x + e^x \cos x) + C.
\end{aligned}$$

Exercises

Use integration by parts to evaluate the following indefinite integrals.

1. $\int t \ln(t+1)\ dt$
2. $\int \frac{(\ln x)^2}{x}\ dx$
3. $\int \arccos x\ dx$
4. $\int e^{2x} \sin x\ dx$
5. $\int e^{x} \sin x\ dx$