MATH 3511 **Probability Homework Problems**

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These are homework problems involving probability density functions. The exact assignment (which problems and due-date) will be announced in class.

Here are some integration formulas which you may useful:

$$\int ue^u \, du = (u-1)e^u + C$$
$$\int u^2 e^u \, du = (u^2 - 2u + 2)e^u + C$$
$$\int u^3 e^u \, du = (u^3 - 3u^2 + 6u - 6)e^u + C$$
$$\int \sin^2 u \, du = \frac{1}{2}(u - \sin u \cos u) + C$$
$$\int \cos^2 u \, du = \frac{1}{2}(u + \sin u \cos u) + C.$$

1. Random variables. Problems are from: *Elementary Statistics*, Mario F. Triola, 8th edition, Addison Wesley Longman, Inc, 2001.

   (a) Identify the given random variable as being discrete or continuous.

      i. The weight of the cola in a randomly selected can.

      ii. The cost of a randomly selected can of Coke.

      iii. The time it takes to fill a can of Pepsi.

      iv. The amount of cola (in ounces) in a can of Pepsi.

      v. The number of cans of Coke on a beverage delivery truck.

   (b) Determine whether a probability distribution is given. In those cases where a probability distribution is not described, identify the requirements that are not satisfied. In those cases where a probability distribution is described, find its mean and standard deviation.

      i. **Gender Selection** In a sudy of the MicroSort gender selection method, couples in a control group are not given treatment, and they each have three children. The probability distribution for the number of girls is given in the table:

x	P(x)
0	0.125
1	0.375
2	0.375
3	0.125

ii. **Overbooked Flights** Air America has a policy of routinely overbooking flights, because past experience shows that some passengers fail to show. The random variable x represents the number of passengers who cannot be boarded because there are more passengers than seats.

x	P(x)
0	0.805
1	0.113
2	0.057
3	0.009
4	0.002

iii. **Gender Bias in Media** A study of gender bias in media coverage involves the selection of people appearing as the subjects in network TV evening news shows. The subjects are randomly selected in groups of four, and the numbers of women are recorded. The probabilities of getting 0, 1, 2, 3, 4 women are 0.334, 0.421, 0.200, 0.042, and 0.003, respectively (based on data from *USA Today*).

(c) **Finding Expected Value for a Magazine Sweepstakes** Reader's Digest ran a sweepstakes in which prizes were listed along with the chances of winning: \$5,000,000 (1 chance in 201,000,000), \$150,000 (1 chance in 201,000,000), \$100,000 (1 chance in 201,000,000), \$25,000 (1 chance in 100,500,000), \$10,000 (1 chance in 50,250,000), \$5,000 (1 chance in 25,125,000), \$200 (1 chance in 8,040,000), \$125 (1 chance in 1,005,000), and a watch valued at \$89 (1 chance in 3774). Find the expected value of the amount won for one entry. Find the expected value if the cost of entering this sweepstakes is the cost of a postage stamp.

(d) **Number of Games in a Baseball World Series** Based on past results found in the *Information Please Almanac*, there is a 0.120 probability that a baseball World Series contest will last four games, a 0.253 probability that a baseball World Series contest will last five games, a 0.217 probability that a baseball World Series contest will last six games, a 0.410 probability that a baseball World Series contest will last seven games. Find the mean and standard deviation for the numbers of games that World Series contests lasts. Is it unusual for a team to "sweep" by winning in four games?

(e) **Rolling Dice** Consider the procedure of rolling a pair of dice 5 times and letting the random variable x represent the number of times that 7 occurs. The following table describes the probability distribution.

x	P(x)
0	0.402
1	0.402
2	?
3	0.032
4	0.003
5	0.000

Find the value of the missing probability. Would it be unusual to roll a pair of dice five times and get at least three 7s? Why or why not?

2. Normal Distribution. Problems are slight modifications of ones given in: *Elementary Statistics*, Mario F. Triola, 8th edition, Addison Wesley Longman, Inc, 2001.

   (a) The heights of adult women in the US are given by the normal distribution $\mu = 63.8$ inches, $\sigma = 2.5$. For adult men: $\mu = 68.5$ inches, $\sigma = 2.8$. What is your height? Are you within one standard deviation, more than one but within two, or more than two? My height is 5'10". Am I "unusual" in the sense that I'm more than two standard deviations from the mean? What would someone's height be on the short side if she were the same standard deviation from the mean?

   (b) **Height Requirement for Women Soldiers**. Assume the heights of women are normally distributed with a mean given by $\mu = 63.6$ in. and a standard deviation given by $\sigma = 2.5$ in. (based on data from the National Health Survey). The U.S. Army requires women's heights to be between 58 and 80 in. Find the percentage of women meeeting that height requirement. Are many women being denied the opportunity to join the Army because they are too short or too tall?

   (c) **Marine Corps Height Requirements** The U.S. Marine Corps requires that men have heights between 64 in. and 78 in. Find the percentage of men meeting those height requirements. (The National Health Survey shows that heights of men are normally distributed with a mean of 69.0 in. and a standard deviation of 2.8 in.) Do you think too many men denied the opportunity to join the Marines because they are too short or too tall?

   (d) **Finding Percentage of Pepsi Cans Under 12 oz** Cans of regular Pepsi are labeled as containing 12 oz. Assume that the actual contents are normally distributed with a mean of 12.29 oz and a standard deviation of 0.09 oz. What percentage of cans contain less than the 12 oz printed on the label?

   (e) **Finding Percentage of Coke Cans Under 12 oz** Cans of regular Coke are labeled as containing 12 oz. Assume that the actual contents are normally distributed with a mean of 12.19 oz and a standard deviation of 0.11 oz. What percentage of cans contain less than the 12 oz printed on the label?

   (f) **Babies at Risk** Weights of newborn babies in the United States are normally distributed with a mean of 3420 g and a standard deviation of 495 g (based on data from “Birth Weight and prenatal Mortality”, by Wilcox et al., *Journal of teh American Medical Association*, Vol. 273, No. 9). A newborn weighing less than 2200 g is considered to be at risk, because the mortality rate for this group is at least 1%. What percentage of newborn babies are in the “at-risk” category? If the Chicago General Hospital has 900 births in a year, how many of the babies are in the “at-risk” category?

3. Find the complex conjugate of Ψ, Ψ^*, and calculate $\Psi^*\Psi$.

   (a) $\Psi = 2 + 3i$

   (b) $\Psi = a + bi$

   (c) $\Psi = e^{-x}\cos x + y$

   (d) $\Psi = e^x i$

   (e) $\Psi = \cos(\theta) + i\sin(\theta)$

   (f) $\Psi = e^{(\pi/3)i}$

   (g) $\Psi = e^{-x^2+iy}$

4. Given the wave function Ψ, find the associated probability function. Here $0 \le x \le L$.

$$\Psi = e^{i\pi x/L} - e^{-i\pi x/L}$$

5. Consider the wave function given by (particle in a box):

$$\Psi = \sin\frac{\pi x}{2}\sin\frac{\pi y}{3}\sin\frac{\pi z}{4}$$

   where $0 \le x \le 2$, $0 \le y \le 3$, and $0 \le z \le 4$.

   (a) Find the associated probability function.

   (b) Find the mean position of the y-coordinate *Set up the integral only; DO NOT SOLVE.*

   (c) What is the probability the molecule is between $0 \le x \le 1$ and $2 \le y \le 3$ and $0 \le z \le 1$?

   (d) What is the probability that $0 \le x \le 1$?

   (e) What is the most likely y-coordinate?

6. One of the electrons in the 2p orbital has a probability density (in spherical coordinates, centered at the nucleus)

$$f(\rho,\phi,\theta) = \frac{1}{32\pi}\frac{\rho^2}{a_o^5}\cos^2\phi e^{-\rho/a_o}, \qquad 0 \le \rho < \infty, \quad 0 \le \phi \le \pi, \quad 0 \le \theta < 2\pi.$$

(a) Verify that this is normalized correctly (write down the integral completely. You may use technology to evaluate, but I would expect that you could do it by hand given $\int u^n e^u \, du$).

(b) Find the most likely (most probable) radius.

(c) Find the most likely angle(s).

(d) Find the average (mean) radius, ρ_{mean}. Compare with the mean radius of an electron in the 1s orbital (in class). Does your answer make sense? Explain.

(e) What is the probability of finding the electron in the region $0 \leq \rho \leq \rho_{mean}$? Does this make sense? Explain.

7. The wave function $\Psi(\theta)$ for the motion of a particle in a ring is of the form

$$\Psi = N e^{in\theta}$$

where n may take on values of 0, 1, 2, 3,...

(a) What is the probability function? What does this mean?

(b) Graph the real part of the function Ψ for $n = 0$, 1, and 2. Physically, why is $\Psi(\theta = 0) = \Psi(\theta = 2\pi)$ for each case?

Solutions

1a) i) continuous, ii) discrete, iii) continuous, iv) continuous, v) discrete.

1b) i) $\mu = 1.5$, $\sigma = 0.9$

1b) ii) Not a probability distribution because $\sum P(x) = 0.9686 \neq 1$ 1b) iii) $\mu = 1.0$, $\sigma = 0.9$

1c) \$.05, \$.05 minus cost of postage stamp.

1d) $\mu = 5.9$, $\sigma = 1.1$ No because the probability of 4 games is .120, which is high (greater than 0.05).

1e) 0.161, Yes, because the probability of at least three 7s is 0.035, which is low (less than 0.05).

2b) 98.74%

2c) 96.32%. I think no.

2d) 0.06%

2e) 4.18%

2f) 0.69%; around 6

3a) $\Psi^* = 2 - 3i$, $\Psi^*\Psi = 13$

3b) $\Psi^* = a - bi$, $\Psi^*\Psi = a^2 + b^2$

3c) $\Psi^* = e^{-x}\cos x + y$, $\Psi^*\Psi = e^{-2x}\cos^2 x + 2ye^{-x}\cos x + y^2$

3d) $\Psi^* = -e^x i$, $\Psi^*\Psi = e^{2x}$

3e) $\Psi^* = \cos(\theta) - i\sin(\theta)$, $\Psi^*\Psi = 1$

3f) $\Psi^* = (\cos(\pi/3) + i\sin(\pi/3))^* = \cos(\pi/3) - i\sin(\pi/3) = e^{-(\pi/3)i}$, $\Psi^*\Psi = 1$

3g) $\Psi^* = (e^{-x^2}e^{iy})^* = e^{-x^2}e^{-iy}$, $\Psi^*\Psi = e^{-2x^2}$

4) $p(x) = \Psi^*\Psi$ *normalized*. Recall that $\int_{-\infty}^{\infty} p(x)\,dx = 1$. Ans: $p(x) = \frac{1}{L}\left[1 - \cos\left(\frac{2\pi x}{L}\right)\right]$.

5a) $p(x) = \frac{1}{3}\sin^2\frac{\pi x}{2}\sin^2\frac{\pi y}{3}\sin^2\frac{\pi z}{4}$

5b) $\dfrac{1}{3}\displaystyle\int_0^2\int_0^3\int_0^4 y\sin^2\frac{\pi x}{2}\sin^2\frac{\pi y}{3}\sin^2\frac{\pi z}{4}\,dz\,dy\,dx$

5c) $\dfrac{1}{3}\displaystyle\int_0^1\int_2^3\int_0^1 \sin^2\frac{\pi x}{2}\sin^2\frac{\pi y}{3}\sin^2\frac{\pi z}{4}\,dz\,dy\,dx = \frac{1}{3}\left(\frac{1}{2}\right)\left(\frac{1}{2} - \frac{3\sqrt{3}}{8\pi}\right)\left(\frac{1}{2} - \frac{1}{\pi}\right) \approx .008$

5d) $\dfrac{1}{3}\displaystyle\int_0^1\int_0^3\int_0^4 \sin^2\frac{\pi x}{2}\sin^2\frac{\pi y}{3}\sin^2\frac{\pi z}{4}\,dz\,dy\,dx = \frac{1}{2}$

5e) Take the partial derivative of the probability function (found in part a) with respect to y and find the critical points. Pick the critical points for which $0 \le y \le 3$ and which give the maximum value of the probability function. Critical points: $y = 0, \frac{3}{2}, 3, \frac{9}{2}, \ldots$. Ans: $\frac{3}{2}$.